

# VARIATIONAL PROPERTIES AND ORBITAL STABILITY OF STANDING WAVES FOR NLS EQUATION ON A STAR GRAPH

RICCARDO ADAMI, CLAUDIO CACCIAPUOTI, DOMENICO FINCO, AND DIEGO NOJA

**ABSTRACT.** We study standing waves for a model of nonlinear Schrödinger equation on a graph  $\mathcal{G}$ . The graph is obtained joining  $N$  halflines at a vertex, i.e. it is a star graph. At the vertex an interaction occurs described by a boundary condition of delta type with strength  $\alpha \leq 0$  (which includes the free or Kirchhoff case  $\alpha = 0$ ). The nonlinearity is of focusing power type. The dynamics is given by an equation of the form  $i \frac{d}{dt} \Psi_t = H \Psi_t - |\Psi_t|^{2\mu} \Psi_t$ , where  $H$  is the Hamiltonian operator which generates the linear Schrödinger dynamics on the graph. We show the existence of several families of standing waves, i.e. solutions of the form  $\Phi_t = e^{i\omega t} \Psi_\omega$  where  $\Psi_\omega$  is a vector amplitude parametrized by a frequency  $\omega$ ; they exist for every sign of the coupling at the vertex, attractive ( $\alpha < 0$ ) and repulsive ( $\alpha > 0$ ) and for every  $\omega > \frac{\alpha^2}{N^2}$ . The number of such families depends on the number  $N$  of the edges of the graph. Furthermore, we determine the ground states, which are the standing waves whose amplitude minimizes the action on the natural (Nehari) constraint, and order the various families of standing waves according to their increasing action, thus determining a nonlinear spectrum of the problem. Finally, we show that the ground states are orbitally stable for every allowed  $\omega$  if the nonlinearity is subcritical or critical; for supercritical nonlinearities there exists an  $\omega^* > \frac{\alpha^2}{N^2}$  such that the ground states are orbitally stable for  $\frac{\alpha^2}{N^2} < \omega < \omega^*$ . The minimization of the action requires technical subtleties. In particular, we develop a non-trivial adaptation of decreasing rearrangements on a star graph, and obtain a generalization of the Pólya-Szegő inequality for the kinetic energy of a rearranged state.

*Keywords:* quantum graphs, non-linear Schrödinger equation, solitary waves.

*MSC 2010:* 35Q55, 81Q35, 37K40, 37K45.

## 1. INTRODUCTION

In the present paper a rigorous analysis of the stationary behaviour of nonlinear Schrödinger equation (NLS) on a graph is given, beginning from the simplest type of unbounded graph, the star graph. In a previous paper [2] the authors studied the behaviour in time of an asymptotically solitary solution of NLS resident on a single edge of the graph in the far past, and impinging on the vertex with various types of couplings, giving a quantitative analysis of reflection and transmission of the solitary wave after the collision at the junction. Here we concentrate on a different phenomenon, namely the existence of persistent nonlinear bound states on the graph (localized, or pinned nonlinear modes), and on their orbital stability, when an attractive interaction is present at the vertex.

Let us briefly give a collocation of the model in the physical context. Generally speaking, one can consider the NLS as a paradigm for the behaviour of nonlinear dispersive equations, but it is also an ubiquitous model appearing in several concrete physical situations. The main fields of application which we have in mind are the propagation of electromagnetic pulses in nonlinear media (typically laser beams in Kerr media or signal propagation in optical fibers), and dynamics of Bose-Einstein condensates (BEC). We are interested in the way solutions of NLS are affected by the presence of inhomogeneities of various type. The propagation on the line in the presence of defects has been a

subject of intense study in the last years and it gives rise to quite interesting phenomena, such as defect induced modes [13, 17, 7], i.e. standing solutions strongly localized around the defect. The presence of defect modes affects propagation by allowing trapping of wave packets, as experimentally shown in the case of local photonic potentials in [28]. On the other hand, nonlinearity can induce escaping of solitons from confining potentials, as demonstrated in [31]. A last interesting phenomenon is the strong alteration of tunneling through potential barriers in the presence of nonlinear defocusing optical media [34]. In this paper we consider NLS propagation through junctions in networks. For example, when the dynamics of a BEC takes place in essentially one-dimensional substrates (“cigar shaped” condensates) or a laser pulse propagates in optical fibers and thin waveguides, the question arises of the effect of a ramified junction on propagation and on the possible generation of stable bound states. The analysis of the behavior of NLS on networks is not yet a fully developed subject, but it is currently growing. Concerning situations of direct physical interest we mention the analysis of scattering at Y junctions (“beam splitters”) and other network configurations (“ring interferometers”) for one dimensional Bose liquids discussed in [33]. Some more results are known for the discrete chain NLS model (DNLS), see in connection with the present paper the analysis in [29]. Other recent developments are in [18, 32]. In particular, in the paper [18] scattering from a complex network sustaining nonlinear Schrödinger dynamics is studied in relation to characterization of quantum chaos.

With these phenomenological and analytical premises in mind we would like to construct a mathematical model capable to represent, in a schematic but rigorous way, the propagation and stationary behaviour of a nonlinear Schrödinger field at a junction of a network. We begin to give the needed preliminaries to rigorously define our model. We recall that the *linear* Schrödinger equation on graphs has been for a long time a very developed subject due to its applications in quantum chemistry, nanotechnologies and more generally mesoscopic physics. Standard references are [10, 11, 24, 25, 23], where more extensive treatments are given. Here we recall only the definitions needed to have a self-contained exposition. We consider a graph  $\mathcal{G}$  constituted by  $N$  infinite half lines attached to a common vertex. The natural Hilbert space where to pose a Schrödinger dynamics is then  $L^2(\mathcal{G}) = \bigoplus_{j=1}^N L^2(\mathbb{R}^+)$ . Elements in  $L^2(\mathcal{G})$  will be represented as function vectors with components in  $L^2(\mathbb{R}^+)$ , namely

$$\Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix}.$$

We denote the elements of  $L^2(\mathcal{G})$  by capital Greek letters, while functions in  $L^2(\mathbb{R}^+)$  are denoted by lowercase Greek letters. We say that  $\Psi$  is symmetric if  $\psi_k$  does not depends on  $k$ . The norm of  $L^2$ -functions on  $\mathcal{G}$  is naturally defined by

$$\|\Psi\|_{L^2(\mathcal{G})}^2 := \sum_{j=1}^N \|\psi_j\|_{L^2(\mathbb{R}^+)}^2.$$

From now on for the  $L^2$ -norm on the graph we drop the subscript and simply write  $\|\cdot\|$ . Accordingly, we denote by  $(\cdot, \cdot)$  the scalar product in  $L^2(\mathcal{G})$ .

Analogously, given  $1 \leq r \leq \infty$ , we define the space  $L^r(\mathcal{G})$  as the set of functions on the graph whose components are elements of the space  $L^r(\mathbb{R}^+)$ , and the norm is correspondingly defined by

$$\|\Psi\|_r^r = \sum_{j=1}^N \|\psi_j\|_{L^r(\mathbb{R}^+)}^r, \quad 1 \leq r < \infty, \quad \|\Psi\|_\infty = \max_{1 \leq j \leq N} \|\psi_j\|_{L^\infty(\mathbb{R}^+)}.$$

Besides, we need to introduce the spaces

$$H^1(\mathcal{G}) \equiv \bigoplus_{j=1}^N H^1(\mathbb{R}^+) \quad H^2(\mathcal{G}) \equiv \bigoplus_{j=1}^N H^2(\mathbb{R}^+),$$

equipped with the norms

$$\|\Psi\|_{H^1}^2 = \sum_{i=1}^N \|\psi_i\|_{H^1(\mathbb{R}^+)}^2, \quad \|\Psi\|_{H^2}^2 = \sum_{i=1}^N \|\psi_i\|_{H^2(\mathbb{R}^+)}^2. \quad (1.1)$$

Whenever a functional norm refers to a function defined on the graph, we omit the symbol  $\mathcal{G}$ .

When an element of  $L^2(\mathcal{G})$  evolves in time, we use in notation the subscript  $t$ : for instance,  $\Psi_t$ . Sometimes we shall write  $\Psi(t)$  in order to highlight the dependence on time, or whenever such a notation is more understandable.

The dynamics we want to set on the graph is generated by a linear part and a nonlinear one. We begin by describing the linear part.

Fixed  $\alpha \in \mathbb{R}$ , we consider a Hamiltonian operator, denoted by  $H$  and called  $\delta$  graph or  $\delta$  vertex, defined on the domain  $\mathcal{D}(H)$

$$\mathcal{D}(H) := \{\Psi \in H^2(\mathcal{G}) \text{ s.t. } \psi_1(0) = \dots = \psi_N(0), \sum_{i=1}^N \psi'_i(0) = \alpha \psi_1(0)\}, \quad (1.2)$$

where  $\psi'_i$  denotes the derivative of the function  $\psi_i$  with respect to the space variable related to the  $i$ -th edge. The action of the operator  $H$  is given by

$$H\Psi = \begin{pmatrix} -\psi''_1 \\ \vdots \\ -\psi''_N \end{pmatrix}.$$

The Hamiltonian  $H$  is a selfadjoint operator on  $L^2(\mathcal{G})$  ([23]) and generalizes to the graph the ordinary Schrödinger operator with  $\delta$  potential of strength  $\alpha$  on the line [9]. Similarly to that case, the interaction is encoded in the boundary condition. The case  $\alpha = 0$  in (1.2) plays a distinguished role and it defines what is usually given the name of free or Kirchhoff boundary condition; we will indicate the corresponding operator as  $H^0$ . Notice that for a graph with two edges, i.e. the line, continuity of wavefunction and its derivative for an element of  $\mathcal{D}(H^0)$  makes the interaction disappear; this fact justifies the name of free Hamiltonian. A  $\delta$  vertex with  $\alpha < 0$  can be interpreted as the presence of a deep attractive potential well or attractive defect. This interpretation can be enforced by showing that, as in the case of the line, the operator  $H$  is a norm resolvent limit for  $\epsilon$  vanishing of a scaled Hamiltonian  $H_\epsilon = H^0 + \alpha V_\epsilon$ , where  $V_\epsilon = \frac{1}{\epsilon} V(\frac{x}{\epsilon})$  and  $V$  is a positive normalized potential on the graph (see [11] and reference therein). The attractive character shows in the fact that for every  $\alpha < 0$  a (single) bound state exists for the linear dynamics, with energy  $-\frac{\alpha^2}{N^2}$ . On the contrary, on a Kirchhoff vertex no bound states exist, the spectrum is purely absolutely continuous, but a zero

energy resonance appears. Finally we recall that in the case of repulsive delta interaction  $\alpha > 0$ , which is however of minor interest here, there are not bound states nor zero energy resonances. The quadratic form  $E^{lin}$  associated to  $H$  is defined on the finite energy space

$$\mathcal{E} \equiv \mathcal{D}(E^{lin}) = \{\Psi \in H^1(\mathcal{G}) \text{ s.t. } \psi_1(0) = \dots = \psi_N(0)\}$$

and is given by

$$E^{lin}[\Psi] = \frac{1}{2} \sum_{i=1}^N \int_0^{+\infty} |\psi'_i(x)|^2 dx + \frac{\alpha}{2} |\psi_1(0)|^2 = \frac{1}{2} \|\Psi'\|^2 + \frac{\alpha}{2} |\psi_1(0)|^2 .$$

The corresponding bilinear form is denoted by  $B(\cdot, \cdot)$  and explicitly given by

$$B(\Psi, \Phi) := \frac{1}{2} \sum_{i=1}^N (\psi_i', \phi_i')_{L^2(\mathbb{R}^+)} + \frac{\alpha}{2} \overline{\psi_1(0)} \phi_1(0). \quad (1.3)$$

As a particular case, the quadratic form  $E^{0,lin}$  associated to  $H^0$  is defined on the same space, that is  $\mathcal{D}(E^{0,lin}) = \mathcal{E}$ , and reads

$$E^{0,lin}[\Psi] = \frac{1}{2} \sum_{i=1}^N \int_0^{+\infty} |\psi'_i(x)|^2 dx = \frac{1}{2} \|\Psi'\|^2 .$$

Now let us introduce the nonlinearity. To this end we define  $G = (G_1, \dots, G_N) : \mathbb{C}^n \rightarrow \mathbb{C}^n$  where  $G$  acts “componentwise” as  $G_i(\zeta) = g(|\zeta_i|)\zeta_i$  for a suitable  $g : \mathbb{R}^+ \rightarrow \mathbb{R}$  and  $\zeta = (\zeta_i) \in \mathbb{C}^n$ .

We are interested in the special but important case of a power nonlinearity of focusing type, so we choose  $g(z) = -|z|^{2\mu}$ ,  $\mu > 0$ .

After this preparation it is well defined the NLS equation on the graph,

$$i \frac{d}{dt} \Psi_t = H \Psi_t - |\Psi_t|^{2\mu} \Psi_t \quad (1.4)$$

where  $\mu > 0$ . This abstract nonlinear Schrödinger equation amounts to a system of scalar NLS equations on the halfline, coupled through the boundary condition at the origin included in the domain (1.2).

In Section 2 we show that for  $\mu > 0$  well-posedness of the dynamics described by equation (1.4) (in weak form) for initial data in the finite energy space  $\mathcal{E}$  holds true. Moreover, if  $0 < \mu < 2$  then the solution exists for all times and blow-up does not occur. Finally, as in the standard NLS on the line, mass  $M(\Psi) = \frac{1}{2} \|\Psi\|^2$  and energy  $E[\Psi]$  are conserved, where

$$E[\Psi] = \frac{1}{2} \|\Psi'\|^2 - \frac{1}{2\mu+2} \|\Psi\|_{2\mu+2}^{2\mu+2} + \frac{\alpha}{2} |\psi_1(0)|^2$$

and analogously, in the case  $\alpha = 0$ , for the Kirchhoff energy

$$E^0[\Psi] = \frac{1}{2} \|\Psi'\|^2 - \frac{1}{2\mu+2} \|\Psi\|_{2\mu+2}^{2\mu+2} .$$

After setting the model and its well posedness (see Section 2), we turn to the main subject of this paper, existence and properties of standing wave solutions to (1.4). Standing waves are solutions of the form

$$\Psi_t(x) = e^{i\omega t} \Psi_\omega(x) .$$

The function  $\Psi_\omega$  is the amplitude or the profile (with some abuse of interpretation) of the standing wave, and we will frequently refer to the set of  $\Psi_\omega$  as to the stationary states of the problem. The amplitude  $\Psi_\omega$  satisfies the *stationary equation*

$$H\Psi_\omega - |\Psi_\omega|^{2\mu}\Psi_\omega = -\omega\Psi_\omega, \quad \omega > 0.$$

This equation has a variational structure.

Let us define the *action* functional

$$S_\omega[\Psi] = E[\Psi] + \omega M[\Psi] = \frac{1}{2}\|\Psi'\|^2 + \frac{\omega}{2}\|\Psi\|^2 - \frac{1}{2\mu+2}\|\Psi\|_{2\mu+2}^{2\mu+2} + \frac{\alpha}{2}|\psi_1(0)|^2.$$

The Euler-Lagrange equation of the action is the stationary equation above. The action  $S_\omega$ , defined on the form domain  $\mathcal{E}$  of the operator  $H$ , is unbounded from below. Nevertheless, it is bounded on the so called natural (or Nehari) constraint  $\{\Psi \in \mathcal{E} \text{ s.t. } I_\omega[\Psi] = 0\}$ , where  $I_\omega[\Psi] = \|\Psi'\|^2 - \|\Psi\|_{2\mu+2}^{2\mu+2} + \omega\|\Psi\|^2 + \alpha|\psi_1(0)|^2$ . Note that  $I_\omega(\Psi_\omega) = S'_\omega(\Psi_\omega)\Psi_\omega$ , and thus the Nehari manifold is a codimension one constraint which contains all the solutions to the stationary equation. One of our main results is the following theorem.

**Theorem 1** (Existence of minimizers for the Action functional). *Let  $\mu > 0$ . There exists  $\alpha^* < 0$  such that for  $-N\sqrt{\omega} < \alpha < \alpha^*$  the action functional  $S_\omega$  constrained to the Nehari manifold admits an absolute minimum, i.e. a  $\Psi \neq 0$  such that  $I_\omega[\Psi] = 0$  and  $S_\omega[\Psi] = \inf \{S_\omega[\Phi] : I_\omega[\Phi] = 0\}$ .*

So the action admits a constrained minimum on the natural constraint for every  $\omega > \frac{\alpha^2}{N^2}$  if the strength  $\alpha$  of the  $\delta$  interaction at the vertex is negative and sufficiently strong. The strategy of the proof, which is a consequence of results of Section 3 and Section 4, makes use of non trivial elements and we give here some remarks. To get the existence of the minimum one has at a certain point to compare the action  $S_\omega$  with  $\alpha < 0$  with the Kirchhoff action  $S_\omega^0$ .

In Section 3 we prove that the Kirchhoff action, while bounded from below on its natural constraint, has no minimum (see [3] for an analogous phenomenon affecting the constrained energy functional). As a matter of fact, the infimum can be exactly computed and it is achieved as the limit over a sequence of functions which escape at infinity on a single edge. A main step in establishing the previous picture and in applying it to the  $\delta$  case, is the exact calculation of the infimum and the identification of the minimizers of the free action; to this end one exploits an extension and generalization of the classical properties of symmetric rearrangements of  $L^p$  and  $H^1$  functions to the case of graphs.

In Section 4 we prove Theorem 1. The analysis follows in part proofs of similar results for singular interactions on the line given in [17, 6], with major modifications due to the fact that in this case the comparison with the free case is not standard. In particular the upper bound in  $\alpha$  given in the statement of Theorem 1 is a consequence of the fact that one needs the condition  $\inf S_\omega < \inf S_\omega^0$  to guarantee the existence of an absolute minimum in the constrained action, and this to penalize situations analogous to escaping minimizers of the free action. A sufficiently strong attractive interaction at the vertex allows to satisfy the previous condition.

We conjecture that the action has a *local* constrained minimum that is larger than the infimum when the condition on  $\alpha$  fails, but presently we do not have a proof of this fact.

In Section 5 an explicit construction of all the stationary states of the problem is obtained, by solving the stationary equation for every value of  $\alpha$ . It turns out that for every  $N$  and  $\omega > \frac{\alpha^2}{N^2}$  there exist families  $\{\Psi_{\omega,j}\}$  of stationary states of different action and energy, which can be ordered in  $j$  to

form a nonlinear spectrum (the family is unique only in the case  $N = 2$ , i.e. the line). The state of minimal action is the ground state  $\Psi_{\omega,0}$ , which is of course the solution to the constrained minimum problem for the action just discussed. The others are excited states, and they exist, for every  $j$ , when  $\omega > \frac{\alpha^2}{(N-2j)^2}$ . See Theorem 4 for a complete description.

Finally, in Section 6 we study the stability of ground states. Stability is an important requisite of a standing wave, because at a physical level unstable states are rapidly dominated by dispersion, drift or blow-up and so are undetectable (instability of NLS with a  $\delta$  potential on the line is studied, partly numerically, in [26]). The concept of stability, due to gauge or  $U(1)$  invariance of the action, is orbital stability. The solutions remain close to the orbit  $e^{i\theta}\Psi_{\omega,0}$  of the ground state for all times if they start close enough to it. The framework in which we study orbital stability of the ground state is the mainstream of Weinstein and Grillakis-Shatah-Strauss theory, which applies to infinite dimensional Hamiltonian systems such as abstract NLS equation when a regular branch of standing waves  $\omega \mapsto \Psi_{\omega,0}$  (not necessarily ground states) exists, which is our case.

According to this theory, to guarantee orbital stability one needs to verify a set of spectral conditions on the linearization of the NLS around the ground state, and a slope (or Vakhitov-Kolokolov in the physical literature) condition concerning the behavior in  $\omega$  of the  $L^2$ -norm of the ground state. Some adaptation of standard methods is needed to treat the singular character of the interaction at the vertex, but in fact it turns out that in the range over which it is valid the previous theorem on the variational characterization of the ground state  $\Psi_{\omega,0}$ , the spectral conditions and the slope condition are encountered for every  $\omega$  and every nonlinearity in the range  $\mu \in (0, 2]$ .

**Theorem 2** (Orbital stability of the ground state). *Let  $\mu \in [0, 2]$ ,  $\alpha < \alpha^* < 0$ ,  $\omega > \frac{\alpha^2}{N^2}$ . Then the ground state  $\Psi_{\omega,0}$  is orbitally stable in  $\mathcal{E}$ .*

The proof of this result is contained in Section 6. Notice that one has orbital stability of the ground state in a range of nonlinearities which includes the critical case. An analogous phenomenon occurs in the case of the line, previously treated in [26]. This marks a difference with the case of a free ( $\alpha = 0$ ) NLS on the line, where one has orbital instability in the critical case. Finally, the proof of the previous theorem (see Remark 6.1) shows that for supercritical nonlinearities  $\mu > 2$  the ground state is orbitally stable for not too large  $\omega$ : there exists a threshold  $\omega^* > \frac{\alpha^2}{N^2}$  such that one has orbital stability for the ground state  $\Psi_{\omega,0}$  with  $\frac{\alpha^2}{N^2} < \omega < \omega^*$  and orbital instability in the opposite case.

Appendix A contains a theory of symmetric rearrangements on star graphs, which is a technical novelty. More precisely, the classical inequalities stating conservation of  $L^p$  norms and domination of kinetic energy are proved. This last property, i.e. the Pólya-Szegő inequality, is particularly interesting because it changes with respect to the case of the line through the presence of a factor which takes into account the number of edges of the graph, and this fact is crucial in the previously described analysis of action minimization on a star graph. We stress the fact that the theory of rearrangements so constructed is a general tool and it is in principle applicable to more general or different problems.

We end this introduction with a few open problems and future directions of study. Concerning technical issues, a different strategy from the one here pursued in the analysis of ground states and their stability is minimization of energy at constant mass (see the classical paper [15] and for models related to the present one [8]); it requires a non trivial extension to graphs of concentration-compactness method and will be studied in a forthcoming paper [1]. Nothing is known up to now

about stability properties of the branches of excited states  $\Psi_{\omega,j}$ , which exist for every  $N > 2$  and sufficiently high  $\omega$ ; this is a subject of special interest because there are only few cases where excited states of NLS equations are explicitly known. The authors plan to study this issue in a subsequent paper. Finally it would be interesting, and perhaps a difficult task, the extension of the analysis here given to different classes of graphs, possibly with non trivial topology. This will need the development of new technical tools, both concerning variational analysis and stability properties.

**Acknowledgments** The authors would like to thank Reika Fukuizumi for several discussions. R. A. was partially supported by the PRIN2009 grant *Critical Point Theory and Perturbative Methods for Nonlinear Differential Equations*.

## 2. WELL-POSEDNESS OF THE MODEL

For our purposes it is sufficient to prove that the solution of the Schrödinger equation is uniquely defined in time in the energy domain and that energy and mass are conserved quantities. This section is devoted to the proof of these conservation laws and of the well-posedness of equation (1.4). In fact along the proofs we shall always work with the weak form of (1.4), namely

$$\Psi_t = e^{-iHt}\Psi_0 + i \int_0^t e^{-iH(t-s)} |\Psi_s|^{2\mu} \Psi_s ds. \quad (2.1)$$

We consider the problem of the well-posedness in the sense of, e.g., [14], i.e., we prove existence and uniqueness of the solution to equation (2.1) in the energy domain of the system. Such a domain turns out to coincide with the form domain of the linear part of equation (1.4). We follow the traditional line of proving first local well posedness, and then extending it to all times by means of a priori estimates provided by the conservation laws. Proceeding as in [2] where the cubic NLS is treated, we show the well-posedness of the dynamics for any  $\mu > 0$ , i.e. local existence and uniqueness for initial data in the energy space. Moreover, we will prove that if  $0 < \mu < 2$ , then the well-posedness is global, i.e. the solution exists for all times and no collapse occurs. For a more extended treatment of the analogous problem for a two-edge vertex (namely, the real line with a point interaction at the origin) see [5].

We endow the energy domain  $\mathcal{E}$  with the  $H^1$ -norm defined in (1.1). Moreover we denote by  $\mathcal{E}^*$  the dual of  $\mathcal{E}$ , i.e. the set of the continuous linear functionals on  $\mathcal{E}$ . We denote the dual product of  $\Gamma \in \mathcal{E}^*$  and  $\Psi \in \mathcal{E}$  by  $\langle \Gamma, \Psi \rangle$ . In such a bracket we sometimes exchange the place of the factor in  $\mathcal{E}^*$  with the place of the factor in  $\mathcal{E}$ : indeed, the duality product follows the same algebraic rules of the standard scalar product.

As usual, one can extend the action of  $H$  to the space  $\mathcal{E}$ , with values in  $\mathcal{E}^*$ , by

$$\langle H\Psi_1, \Psi_2 \rangle := B[\Psi_1, \Psi_2],$$

where  $B[\cdot, \cdot]$  denotes the bilinear form associated to the selfadjoint operator  $H$ . Furthermore, for any  $\Psi \in \mathcal{E}$  the identity

$$\frac{d}{dt} e^{-iHt} \Psi = -iH e^{-iHt} \Psi \quad (2.2)$$

holds in  $\mathcal{E}^*$  too. To prove it, one can first test the functional  $\frac{d}{dt}e^{-iHt}\Psi$  on an element  $\Xi$  in the operator domain  $\mathcal{E}$ , obtaining

$$\left\langle \frac{d}{dt}e^{-iHt}\Psi, \Xi \right\rangle = \lim_{h \rightarrow 0} \left( \Psi, \frac{e^{iH(t+h)}\Xi - e^{iHt}\Xi}{h} \right) = (\Psi, iHe^{iHt}\Xi) = \langle -iHe^{-iHt}\Psi, \Xi \rangle.$$

Then, the result can be extended to  $\Xi \in \mathcal{E}$  by a density argument.

Besides, by (2.2), the differential version (1.4) of the Schrödinger equation holds in  $\mathcal{E}^*$ .

In order to prove a well-posedness result we need to generalize standard one-dimensional Gagliardo-Nirenberg estimates to graphs, i.e.

$$\|\Psi\|_p \leq C \|\Psi'\|^{\frac{1}{2} - \frac{1}{p}} \|\Psi\|^{\frac{1}{2} + \frac{1}{p}}, \quad (2.3)$$

where the  $C > 0$  is a positive constant which depends on the index  $p$  only. The proof of (2.3) follows immediately from the analogous estimates for functions of the real line, considering that any function in  $H^1(\mathbb{R}^+)$  can be extended to an even function in  $H^1(\mathbb{R})$ , and applying this reasoning to each component of  $\Psi$  (see also [30, I.31]).

**Proposition 2.1** (Local well-posedness in  $\mathcal{E}$ ).

Let  $\mu > 0$ . For any  $\Psi_0 \in \mathcal{E}$ , there exists  $T > 0$  such that the equation (2.1) has a unique solution  $\Psi \in C^0([0, T], \mathcal{E}) \cap C^1([0, T], \mathcal{E}^*)$ . Moreover, eq. (2.1) has a maximal solution  $\Psi^{\max}$  defined on an interval of the form  $[0, T^*)$ , and the following “blow-up alternative” holds: either  $T^* = \infty$  or

$$\lim_{t \rightarrow T^*} \|\Psi_t^{\max}\|_{\mathcal{E}} = +\infty,$$

where we denoted by  $\Psi_t^{\max}$  the function  $\Psi^{\max}$  evaluated at time  $t$ .

*Proof.* We define the space  $\mathcal{X} := L^\infty([0, T], \mathcal{E})$ , endowed with the norm  $\|\Psi\|_{\mathcal{X}} := \sup_{t \in [0, T]} \|\Psi_t\|_{\mathcal{E}}$ . Given  $\Psi_0 \in \mathcal{E}$ , we define the map  $G : \mathcal{X} \rightarrow \mathcal{X}$  as

$$G\Phi := e^{-iH \cdot} \Psi_0 + i \int_0^\cdot e^{-iH(\cdot-s)} |\Phi_s|^{2\mu} \Phi_s ds.$$

We first notice that the nonlinearity preserves the space  $\mathcal{E}$ . Then by  $|(|\phi|^{2\mu}\phi)'| \leq C|\phi|^{2\mu}|\phi'|$  and using Hölder and Gagliardo-Nirenberg inequalities, one obtains

$$\| |\Phi_s|^{2\mu} \Phi_s \|_{\mathcal{E}} \leq C \|\Phi_s\|_{\mathcal{E}}^{2\mu+1},$$

so that

$$\|G\Phi\|_{\mathcal{X}} \leq \|\Psi_0\|_{\mathcal{E}} + C \int_0^T \|\Phi_s\|_{\mathcal{E}}^{2\mu+1} ds \leq \|\Psi_0\|_{\mathcal{E}} + CT \|\Phi\|_{\mathcal{X}}^{2\mu+1}. \quad (2.4)$$

Analogously, given  $\Phi, \Xi \in \mathcal{E}$ , one has

$$\|G\Phi - G\Xi\|_{\mathcal{X}} \leq CT (\|\Phi\|_{\mathcal{X}}^{2\mu} + \|\Xi\|_{\mathcal{X}}^{2\mu}) \|\Phi - \Xi\|_{\mathcal{X}}. \quad (2.5)$$

We point out that the constant  $C$  appearing in (2.4) and (2.5) is independent of  $\Psi_0$ ,  $\Phi$ , and  $\Xi$ . Now let us restrict the map  $G$  to elements  $\Phi$  such that  $\|\Phi\|_{\mathcal{X}} \leq 2\|\Psi_0\|_{\mathcal{E}}$ . From (2.4) and (2.5), if  $T$  is chosen to be strictly less than  $(8C\|\Psi_0\|_{\mathcal{E}}^{2\mu})^{-1}$ , then  $G$  is a contraction of the ball in  $\mathcal{X}$  of radius  $2\|\Psi_0\|_{\mathcal{E}}$ , and so, by the contraction lemma, there exists a unique solution to (2.1) in the time interval  $[0, T)$ . By a standard one-step bootstrap argument one immediately has that the solution actually belongs to  $C^0([0, T], \mathcal{E})$ , and due to the validity of (1.4) in the space  $\mathcal{E}^*$  we immediately have that the solution  $\Psi$  actually belongs to  $C^0([0, T], \mathcal{E}) \cap C^1([0, T], \mathcal{E}^*)$ .



The proof of the existence of a maximal solution is standard, while the blow-up alternative is a consequence of the fact that, whenever the  $\mathcal{E}$ -norm of the solution is finite, it is possible to extend it for a further time by the same contraction argument.  $\square$

The next step consists in the proof of the conservation laws.

**Proposition 2.2** (Conservation laws).

Let  $\mu > 0$ . For any solution  $\Psi \in C^0([0, T], \mathcal{E}) \cap C^1([0, T], \mathcal{E}^*)$  to the problem (2.1), the following conservation laws hold at any time  $t$ :

$$M[\Psi_t] = M[\Psi_0], \quad E[\Psi_t] = E[\Psi_0].$$

*Proof.* The conservation of the  $L^2$ -norm can be immediately obtained by the validity of equation (1.4) in the space  $\mathcal{E}$ :

$$\frac{d}{dt} M[\Psi_t] = \operatorname{Re} \left\langle \Psi_t, \frac{d}{dt} \Psi_t \right\rangle = 0$$

by the self-adjointness of  $H$ . In order to prove the conservation of the energy, first we notice that  $\langle \Psi_t, H\Psi_t \rangle$  is differentiable as a function of time. Indeed,

$$\frac{1}{h} [\langle \Psi_{t+h}, H\Psi_{t+h} \rangle - \langle \Psi_t, H\Psi_t \rangle] = \left\langle \frac{\Psi_{t+h} - \Psi_t}{h}, H\Psi_{t+h} \right\rangle + \left\langle H\Psi_t, \frac{\Psi_{t+h} - \Psi_t}{h} \right\rangle$$

and then, passing to the limit  $h \rightarrow 0$ ,

$$\frac{d}{dt} \langle \Psi_t, H\Psi_t \rangle = 2 \operatorname{Re} \left\langle \frac{d}{dt} \Psi_t, H\Psi_t \right\rangle = -2 \operatorname{Im} \langle |\Psi_t|^{2\mu} \Psi_t, H\Psi_t \rangle, \quad (2.6)$$

where we used the self-adjointness of  $H$  and (1.4). Furthermore,

$$\frac{d}{dt} (\Psi_t, |\Psi_t|^{2\mu} \Psi_t) = \frac{d}{dt} (\Psi_t^{\mu+1}, \Psi_t^{\mu+1}) = -2(\mu+1) \operatorname{Im} \langle |\Psi_t|^{2\mu} \Psi_t, H\Psi_t \rangle. \quad (2.7)$$

From (2.6) and (2.7) one then obtains

$$\frac{d}{dt} E[\Psi_t] = \frac{1}{2} \frac{d}{dt} \langle \Psi_t, H\Psi_t \rangle - \frac{1}{2\mu+2} \frac{d}{dt} (\Psi_t, |\Psi_t|^2 \Psi_t)_{L^2} = 0$$

and the proposition is proved.  $\square$

**Corollary 2.1** (Global well posedness).

Let  $0 < \mu < 2$ . For any  $\Psi_0 \in \mathcal{E}$ , the equation (2.1) has a unique solution  $\Psi \in C^0([0, \infty), \mathcal{E}) \cap C^1([0, \infty), \mathcal{E}^*)$ .

*Proof.* By estimate (2.3) with  $p = \infty$  and conservation of the  $L^2$ -norm, there exists a constant  $C$ , that depends on  $\Psi_0$  only, such that

$$E[\Psi_0] = E[\Psi_t] \geq \frac{1}{2} \|\Psi'_t\|^2 - C \|\Psi'_t\|^\mu$$

Therefore a uniform (in  $t$ ) bound on  $\|\Psi'_t\|$  is obtained. As a consequence, one has that no blow-up in finite time can occur, and therefore, by the blow-up alternative, the solution is global in time.  $\square$

### 3. VARIATIONAL ANALYSIS: THE KIRCHHOFF VERTEX

In this section we compute the infimum of the action functional for the Kirchhoff case. As often in this framework, the action functional is unbounded from below and we have to restrict it to the Nehari manifold, or natural constraint manifold, in order to have a functional bounded from below. The knowledge of the infimum of the constrained action will be a key ingredient in the next section in the proof of the main theorem.

The strategy of the computation of the infimum is standard: first we derive a lower bound and then we show that this lower bound is optimal by means of a minimizing sequence. In the derivation of the lower bound symmetric rearrangements are used. Using this technique we can map the initial variational problem into a variational problem with symmetric functions which can be reduced to a problem on the halfline providing the required estimate.

The minimizing sequence shows in fact that the constrained action exhibits a sort of spontaneous symmetry breaking in the Kirchhoff case. That is, although the functional is symmetric, the minimizing sequence is localized on a single edge.

As defined in the introduction, in the Kirchhoff case the action functional is given by

$$S_\omega^0[\Psi] = E^0[\Psi] + \omega M[\Psi] = \frac{1}{2} \|\Psi'\|^2 + \frac{\omega}{2} \|\Psi\|^2 - \frac{1}{2\mu+2} \|\Psi\|_{2\mu+2}^{2\mu+2}, \quad (3.1)$$

while the Nehari functional  $I_\omega^0$  reads

$$I_\omega^0[\Psi] = \|\Psi'\|^2 - \|\Psi\|_{2\mu+2}^{2\mu+2} + \omega \|\Psi\|^2.$$

The Nehari manifold is defined by  $\{\Psi \in \mathcal{E}, \Psi \neq 0 \text{ s.t. } I_\omega^0[\Psi] = 0\}$ . The action restricted to the Nehari manifold will be named reduced action and is given by

$$\tilde{S}[\Psi] = S_\omega^0[\Psi] - \frac{1}{2} I_\omega^0[\Psi] = \frac{\mu}{2\mu+2} \|\Psi\|_{2\mu+2}^{2\mu+2}. \quad (3.2)$$

It is understood that the domain of all the functionals is always  $\mathcal{E}$ .

**Theorem 3** (Infimum of the Action for the Kirchhoff case).

*The infimum of the action functional  $S_\omega^0$  restricted to the Nehari manifold is given by:*

$$\inf\{S_\omega^0[\Psi] \text{ s.t. } \Psi \in \mathcal{E}, \Psi \neq 0, I_\omega^0[\Psi] = 0\} \equiv d^0(\omega) = (\mu+1)^{\frac{1}{\mu}} \omega^{\frac{1}{\mu} + \frac{1}{2}} \int_0^1 (1-t^2)^{\frac{1}{\mu}} dt. \quad (3.3)$$

#### Proof

The proof of (3.3) is divided into two parts: first we derive a lower bound for  $S_\omega^0$ , then we prove that the lower bound is optimal by means of a minimizing sequence.

In order to derive a lower bound, we consider an auxiliary variational problem with symmetric functions. This is done by using the rearrangements on the graph which are discussed in Appendix A.

Let  $\Phi \in \mathcal{E}$  and let  $\Phi^*$  be its symmetric rearrangement. We know that  $\Phi^*$  is positive, symmetric and  $\Phi^* \in \mathcal{E}$ . Moreover, by Theorems 6 and A.1, we have

$$\|\Phi\| = \|\Phi^*\| \quad \|\Phi\|_{2\mu+2} = \|\Phi^*\|_{2\mu+2} \quad \|\Phi'\| \geq \frac{2}{N} \|\Phi^*\|. \quad (3.4)$$

Therefore for  $\Phi \in \mathcal{E}$  such that  $I_\omega^0[\Phi] = 0$  we have

$$\frac{4}{N^2} \|\Phi^{*'}\|^2 - \|\Phi^*\|_{2\mu+2}^{2\mu+2} + \omega \|\Phi^*\|^2 \leq I_\omega^0[\Phi] = 0 \quad (3.5)$$

and

$$\tilde{S}[\Phi] = \tilde{S}[\Phi^*]. \quad (3.6)$$

Taking into account (3.2), and the above properties of  $\Phi^*$  one can enlarge the domain in the following way in order to lower the infimum,

$$\begin{aligned} \inf\{S_\omega^0[\Phi] \text{ s.t. } \Phi \in \mathcal{E}, \Phi \neq 0, I_\omega^0[\Phi] = 0\} &= \inf\{\tilde{S}[\Phi] \text{ s.t. } \Phi \in \mathcal{E}, \Phi \neq 0, I_\omega^0[\Phi] = 0\} \\ &\geq \inf\{\tilde{S}[\Phi] \text{ s.t. } \Phi \in \mathcal{E}, \Phi \neq 0, \Phi \text{ symmetric}, \frac{4}{N^2} \|\Phi'\|^2 - \|\Phi\|_{2\mu+2}^{2\mu+2} + \omega \|\Phi\|^2 \leq 0\}. \end{aligned} \quad (3.7)$$

Under the scaling,  $\Phi(\cdot) \rightsquigarrow \lambda^{1/2} \Phi(\lambda \cdot)$ ,  $\lambda > 0$ , the last variational problem scales as

$$\begin{aligned} \inf\left\{\tilde{S}[\Phi] \text{ s.t. } \Phi \in \mathcal{E}, \Phi \neq 0, \Phi \text{ symmetric}, \frac{4}{N^2} \|\Phi'\|^2 - \|\Phi\|_{2\mu+2}^{2\mu+2} + \omega \|\Phi\|^2 \leq 0\right\} &= \\ \inf\lambda^\mu \left\{\tilde{S}[\Phi] \text{ s.t. } \Phi \in \mathcal{E}, \Phi \neq 0, \Phi \text{ symmetric}, \frac{4}{N^2} \lambda^2 \|\Phi'\|^2 - \lambda^\mu \|\Phi\|_{2\mu+2}^{2\mu+2} + \omega \|\Phi\|^2 \leq 0\right\}. \end{aligned}$$

It is convenient to choose  $\lambda$  as

$$\frac{4}{N^2} \lambda^2 = \lambda^\mu \quad \text{so that} \quad \lambda = \left(\frac{N}{2}\right)^{\frac{2}{2-\mu}}$$

in order to reconstruct a Nehari manifold with a rescaled  $\omega$  as constraint. Moreover due to the symmetry of  $\Phi$  we have

$$\begin{aligned} \left(\frac{N}{2}\right)^{\frac{2\mu}{2-\mu}} \inf\left\{\tilde{S}[\Phi] \text{ s.t. } \Phi \in \mathcal{E}, \Phi \neq 0, \Phi \text{ symmetric}, \|\Phi'\|^2 - \|\Phi\|_{2\mu+2}^{2\mu+2} + \omega \left(\frac{2}{N}\right)^{\frac{2\mu}{2-\mu}} \|\Phi\|^2 \leq 0\right\} \\ = N \left(\frac{N}{2}\right)^{\frac{2\mu}{2-\mu}} \inf\left\{\frac{\mu}{2\mu+2} \|\phi\|_{L^{2\mu+2}(\mathbb{R}^+)}^{2\mu+2} \text{ s.t. } \phi \in H^1(\mathbb{R}^+), \right. \\ \left. \phi \neq 0, \|\phi'\|_{L^2(\mathbb{R}^+)}^2 - \|\phi\|_{L^{2\mu+2}(\mathbb{R}^+)}^{2\mu+2} + \omega \left(\frac{2}{N}\right)^{\frac{2\mu}{2-\mu}} \|\phi\|_{L^2(\mathbb{R}^+)}^2 \leq 0\right\}. \end{aligned} \quad (3.8)$$

It is convenient to introduce a variational problem on the half line and an auxiliary variational problem on the line. Let  $d^{\text{half}}(\omega)$  and  $d^{\text{line}}(\omega)$  be defined in the following way:

$$\begin{aligned} d^{\text{half}}(\omega) &= \inf\left\{\frac{\mu}{2\mu+2} \|\phi\|_{L^{2\mu+2}(\mathbb{R}^+)}^{2\mu+2} \text{ s.t. } \phi \in H^1(\mathbb{R}^+), \phi \neq 0, \|\phi'\|_{L^2(\mathbb{R}^+)}^2 - \|\phi\|_{L^{2\mu+2}(\mathbb{R}^+)}^{2\mu+2} + \omega \|\phi\|_{L^2(\mathbb{R}^+)}^2 \leq 0\right\} \\ d^{\text{line}}(\omega) &= \inf\left\{\frac{\mu}{2\mu+2} \|\phi\|_{L^{2\mu+2}(\mathbb{R})}^{2\mu+2} \text{ s.t. } \phi \in H^1(\mathbb{R}), \phi \neq 0, \|\phi'\|_{L^2(\mathbb{R})}^2 - \|\phi\|_{L^{2\mu+2}(\mathbb{R})}^{2\mu+2} + \omega \|\phi\|_{L^2(\mathbb{R})}^2 \leq 0\right\}. \end{aligned}$$

Notice that the following inequality holds true:

$$2d^{\text{half}}(\omega) \geq d^{\text{line}}(\omega). \quad (3.9)$$

Indeed, by absurd, assume  $2d^{\text{half}}(\omega) < d^{\text{line}}(\omega)$  and let  $\phi_n$  be a minimizing sequence for the problem on the halfline. We can extend  $\phi_n$  by parity and obtain a sequence  $\tilde{\phi}_n \in H^1(\mathbb{R})$  such that

$$\|\tilde{\phi}'_n\|_{L^2(\mathbb{R})}^2 - \|\tilde{\phi}_n\|_{L^{2\mu+2}(\mathbb{R})}^{2\mu+2} + \omega\|\tilde{\phi}_n\|_{L^2(\mathbb{R})}^2 \leq 0 \quad \|\tilde{\phi}_n\|_{L^{2\mu+2}(\mathbb{R})}^{2\mu+2} = 2\|\phi_n\|_{L^{2\mu+2}(\mathbb{R}^+)}^{2\mu+2}.$$

Passing to the limit one would obtain

$$d^{\text{line}}(\omega) \geq \liminf_n \frac{\mu}{2\mu+2} \|\tilde{\phi}_n\|_{L^{2\mu+2}(\mathbb{R})}^{2\mu+2} = 2d^{\text{half}}(\omega)$$

which contradicts our absurd hypothesis. Therefore  $1/2 d^{\text{line}}(\omega)$  provides a lower bound for the variational problem we are interested in. On the other hand the exact expression of  $d^{\text{line}}(\omega)$  can be easily obtained from known results (see [14] Ch. VIII), and it is given by:

$$d^{\text{line}}(\omega) = (\mu+1)^{\frac{1}{\mu}} \omega^{\frac{1}{\mu} + \frac{1}{2}} \int_0^1 (1-t^2)^{\frac{1}{\mu}} dt. \quad (3.10)$$

Taking into account (3.7), (3.8), (3.9) and (3.10) we can conclude

$$\begin{aligned} d^0(\omega) &\geq \frac{N}{2} \left( \frac{N}{2} \right)^{\frac{2\mu}{2-\mu}} (\mu+1)^{\frac{1}{\mu}} \left[ \omega \left( \frac{2}{N} \right)^{\frac{2\mu}{2-\mu}} \right]^{\frac{\mu+2}{2\mu}} \int_0^1 (1-t^2)^{\frac{1}{\mu}} dt \\ &= (\mu+1)^{\frac{1}{\mu}} \omega^{\frac{\mu+2}{2\mu}} \int_0^1 (1-t^2)^{\frac{1}{\mu}} dt. \end{aligned} \quad (3.11)$$

Estimate (3.11) closes the first part of the proof. Now it is sufficient to exhibit a sequence of trial functions  $\Phi_n$  satisfying the constraint and such that  $\tilde{S}[\Phi_n] \rightarrow (\mu+1)^{\frac{1}{\mu}} \omega^{\frac{\mu+2}{2\mu}} \int_0^1 (1-t^2)^{\frac{1}{\mu}} dt$ . We consider a sequence of soliton-like functions escaping to infinity, i.e.

$$(\Phi_n)_i(x) = \begin{cases} \phi_n(x) = \phi_s(x-n)\chi(x) & i = 1 \\ 0 & i \neq 1 \end{cases} \quad (3.12)$$

where  $\phi_s$  is defined in Appendix B by (B.1) and  $\chi$  is a  $C^\infty(\mathbb{R}^+)$  function such that  $0 \leq \chi \leq 1$ ,  $\chi = 0$  for  $0 \leq x \leq 1$  and  $\chi(x) = 1$  for  $x \geq 2$ . The sequence  $\Phi_n$  belongs to  $\mathcal{E}$  but does not satisfy the constraint  $I_\omega^0[\Phi_n] = 0$ . It is straightforward to check that

$$\|\Phi_n\|_{2\mu+2} \geq c \quad (3.13)$$

where the r.h.s. of (3.13) depends on  $\omega$  and  $\mu$ . In the remaining part of the proof we shall not make explicit the dependence on  $\omega$  and  $\mu$  of the constant appearing in estimates. Let  $\delta_n$  be defined by

$$\delta_n = \left( \frac{\|\Phi'_n\|^2 + \omega\|\Phi_n\|^2}{\|\Phi_n\|_{2\mu+2}^{2\mu+2}} \right)^{\frac{1}{2\mu}}. \quad (3.14)$$

It is straightforward to check that  $I^0[\delta_n \Phi_n] = 0$ . Then in order to prove (3.3), it is sufficient to prove that

$$\lim_{n \rightarrow \infty} \tilde{S}[\delta_n \Phi_n] = (\mu+1)^{\frac{1}{\mu}} \omega^{\frac{\mu+2}{2\mu}} \int_0^1 (1-t^2)^{\frac{1}{\mu}} dt. \quad (3.15)$$

Now we prove that

$$\lim_{n \rightarrow \infty} \delta_n = 1. \quad (3.16)$$

We have

$$\delta_n = \left( 1 + \frac{\|\Phi'_n\|^2 + \omega\|\Phi_n\|^2 - \|\Phi_n\|_{2\mu+2}^{2\mu+2}}{\|\Phi_n\|_{2\mu+2}^{2\mu+2}} \right)^{\frac{1}{2\mu}} \quad (3.17)$$

and by (3.13), it is sufficient to prove that

$$\lim_{n \rightarrow \infty} \|\Phi'_n\|^2 + \omega\|\Phi_n\|^2 - \|\Phi_n\|_{2\mu+2}^{2\mu+2} = 0. \quad (3.18)$$

Taking into account (3.12) and (B.2) and integrating by parts one has

$$\begin{aligned} & \left| \|\Phi'_n\|^2 + \omega\|\Phi_n\|^2 - \|\Phi_n\|_{2\mu+2}^{2\mu+2} \right| \leq \\ & c \int_0^\infty |\phi_s(x-n)|^2 |\chi''(x)| |\chi(x)| dx + c \int_0^\infty |\phi_s(x-n)| |\phi'_s(x-n)| |\chi'(x)| |\chi(x)| dx \equiv \mathcal{R}_n. \end{aligned} \quad (3.19)$$

The remainder  $\mathcal{R}_n$  can be estimated using the exponential decay of  $\phi_s$  and  $\phi'_s$  in the following way

$$|\mathcal{R}_n| \leq c \int_{n-1}^\infty |\phi_s(x)|^2 dx + c \int_{n-1}^\infty |\phi_s(x)| |\phi'_s(x)| dx \leq c \int_n^\infty e^{-cx} dx \leq ce^{-cn}. \quad (3.20)$$

This proves (3.16) while (3.15) is reduced to prove that

$$\lim_{n \rightarrow \infty} \tilde{S}[\Phi_n] = (\mu+1)^{\frac{1}{\mu}} \omega^{\frac{\mu+2}{2\mu}} \int_0^1 (1-t^2)^{\frac{1}{\mu}} dt.$$

The last equality follows by dominated convergence and (B.5):

$$\begin{aligned} \lim_{n \rightarrow \infty} \tilde{S}[\Phi_n] &= \lim_{n \rightarrow \infty} \frac{\mu}{2\mu+2} \int_0^\infty |\phi_s(x-n)\chi(x)|^{2\mu+2} dx = \lim_{n \rightarrow \infty} \frac{\mu}{2\mu+2} \int_{-n}^\infty |\phi_s(x)\chi(x+n)|^{2\mu+2} dx = \\ &= \frac{\mu}{2\mu+2} \int_{-\infty}^\infty |\phi_s(x)|^{2\mu+2} dx = (\mu+1)^{\frac{1}{\mu}} \omega^{\frac{2+\mu}{2\mu}} \int_0^1 (1-t^2)^{\frac{1}{\mu}} dt. \end{aligned} \quad (3.21)$$

The proof is concluded. □

The previous proof shows that the infimum  $d^0(\omega)$  is approximated by the action of a soliton escaping to infinity. Moreover notice that the minimizing sequence weakly converges to the vanishing function.

#### 4. VARIATIONAL ANALYSIS: THE $\delta$ VERTEX

In this section we discuss the variational properties of the action functional in the general case with  $\alpha < 0$ . In fact we prove that there exists  $\alpha^* < 0$  such that for  $-N\sqrt{\omega} < \alpha < \alpha^*$  the action functional constrained to the Nehari manifold admits an absolute minimum. The proof of this statement is broken into several lemmas. Firstly, in Lemma 4.1 we prove an equivalent formulation of the variational problem we are studying. Then, in Lemma 4.2 and Proposition 4.1 we prove that the infimum of the constrained action is strictly positive and smaller than the infimum of the Kirchhoff action, therefore for  $\alpha$  negative enough the infimum is not reached by functions escaping at infinity like (3.12), otherwise the two infima would coincide. This is a key ingredient in the proof of the main Theorem 1, where we prove that a minimizing sequence admits subsequences with non trivial weak limit. Finally we prove that this limit is the absolute minimum.

We recall from the introduction the action functional, given by

$$S_\omega[\Psi] = E[\Psi] + \omega M[\Psi] = S_\omega^0[\Psi] + \frac{\alpha}{2} |\psi_1(0)|^2$$

and the Nehari functional

$$I_\omega[\Psi] = \|\Psi'\|^2 - \|\Psi\|_{2\mu+2}^{2\mu+2} + \omega \|\Psi\|^2 + \alpha |\psi_1(0)|^2.$$

The Nehari manifold is given by  $\{\Psi \in \mathcal{E}, \Psi \neq 0, \text{ s.t. } I_\omega[\Psi] = 0\}$ . It is understood that the above functionals are defined on the form domain  $\mathcal{E}$ . The action restricted to the Nehari manifold will be named again reduced action and is defined by

$$\tilde{S}[\Psi] = \frac{\mu}{2\mu+2} \|\Psi\|_{2\mu+2}^{2\mu+2} = S_\omega[\Psi] - \frac{1}{2} I_\omega[\Psi]. \quad (4.1)$$

We introduce also the function

$$d(\omega) = \inf\{S_\omega[\Psi] \text{ s.t. } \Psi \in \mathcal{E}, \Psi \neq 0, I_\omega[\Psi] = 0\}. \quad (4.2)$$

In the following let  $\alpha < 0$ .

We want to prove that for  $\alpha$  smaller than a threshold value  $\alpha^*$  the action  $S_\omega$  constrained to the Nehari manifold admits an absolute minimum.

Firstly we give an equivalent formulation of this variational problem.

**Lemma 4.1.** *The following equality holds*

$$\inf\{S_\omega[\Psi] \text{ s.t. } \Psi \in \mathcal{E}, \Psi \neq 0, I_\omega[\Psi] = 0\} = \inf\{\tilde{S}[\Psi] \text{ s.t. } \Psi \in \mathcal{E}, \Psi \neq 0, I_\omega[\Psi] \leq 0\}. \quad (4.3)$$

Moreover  $\Phi \in \mathcal{E}$  satisfies  $\tilde{S}[\Phi] = d(\omega)$  and  $I_\omega[\Phi] \leq 0$  iff  $S_\omega[\Phi] = d(\omega)$  and  $I_\omega[\Phi] = 0$ .

### Proof

The idea is the following: if a function  $\Phi$  is not on the Nehari manifold and  $I_\omega[\Phi] < 0$ , then by multiplication by a suitable scalar, it can be pulled on the manifold lowering the reduced action at the same time. First notice that by (4.1) we have immediately

$$\inf\{S_\omega[\Psi] \text{ s.t. } \Psi \in \mathcal{E}, \Psi \neq 0, I_\omega[\Psi] = 0\} \geq \inf\{\tilde{S}[\Psi] \text{ s.t. } \Psi \in \mathcal{E}, \Psi \neq 0, I_\omega[\Psi] \leq 0\}, \quad (4.4)$$

since  $S_\omega$  and  $\tilde{S}$  coincide on the Nehari manifold.

Now take  $\Phi \in \mathcal{E}$  such that  $I_\omega[\Phi] < 0$  and define

$$\beta = \left( \frac{\|\Phi'\|^2 + \alpha |\phi_1(0)|^2 + \omega \|\Phi\|^2}{\|\Phi\|_{2\mu+2}^{2\mu+2}} \right)^{\frac{1}{2\mu}}. \quad (4.5)$$

Since  $I_\omega[\Phi] < 0$  then  $\beta < 1$ . Moreover by direct computation one has

$$I_\omega[\beta\Phi] = 0. \quad (4.6)$$

Then, using again (4.1), one has

$$S_\omega[\beta\Phi] = \tilde{S}[\beta\Phi] = \beta^{2\mu+2} \tilde{S}[\Phi] < \tilde{S}[\Phi] \quad (4.7)$$

then

$$\inf\{S_\omega[\Psi] \text{ s.t. } \Psi \in \mathcal{E}, I_\omega[\Psi] = 0\} \leq \inf\{\tilde{S}[\Psi] \text{ s.t. } \Psi \in \mathcal{E}, I_\omega[\Psi] \leq 0\} \quad (4.8)$$

and identity (4.3) has been proved.

Notice that if  $\Phi$  minimizes  $S_\omega$  on  $I_\omega = 0$  then it minimizes also  $\tilde{S}$  on  $I_\omega \leq 0$  by (4.3). Suppose now that  $\tilde{S}[\Phi] = d(\omega)$  and  $I_\omega[\Phi] \leq 0$ . Then defining  $\beta$  as above one has that  $S_\omega[\beta\Phi] < \tilde{S}[\Phi] = d(\omega)$  which is a contradiction to the definition of  $d(\omega)$ .

□

**Lemma 4.2.** *Assume  $\omega > \alpha^2/N^2$ . Then  $d(\omega)$  is strictly positive.*

**Proof**

Firstly we derive an elementary Sobolev inequality for the halfline. Let  $f \in H^1(\mathbb{R})$  and denote by  $\hat{f}(k)$  its Fourier transform. Then, we have

$$|f(0)| \leq \frac{1}{\sqrt{2\pi}} \int |\hat{f}(k)| dk = \frac{1}{\sqrt{2\pi}} \int |\hat{f}(k)| \frac{(ak^2 + a^{-1})}{(ak^2 + a^{-1})} dk. \quad (4.9)$$

By Cauchy-Schwartz inequality we have

$$|f(0)|^2 \leq \frac{1}{2} (a \|f'\|_{L^2(\mathbb{R})}^2 + a^{-1} \|f\|_{L^2(\mathbb{R})}^2). \quad (4.10)$$

If  $\phi \in H^1(\mathbb{R}^+)$  we can extend it by parity to a function on the line and apply (4.10). In this way we finally have

$$|\phi(0)|^2 \leq a \|\phi'\|_{L^2(\mathbb{R}^+)}^2 + a^{-1} \|\phi\|_{L^2(\mathbb{R}^+)}^2. \quad (4.11)$$

Now take  $\Phi \in \mathcal{E}$  then by (4.11), we have

$$|\phi_1(0)|^2 = \frac{1}{N} \sum_{i=1}^N |\phi_i(0)|^2 \leq \frac{1}{N} \sum_{i=1}^N \left( a \|\phi_i\|_{L^2(\mathbb{R}^+)}^2 + \frac{1}{a} \|\phi_i'\|_{L^2(\mathbb{R}^+)}^2 \right) = \frac{a}{N} \|\Phi\|^2 + \frac{1}{aN} \|\Phi'\|^2. \quad (4.12)$$

Then, using again (4.3) and with a suitable choice of  $a$  (it is possible due to the restriction on  $\omega$ ) we have

$$0 \geq \left( 1 - \frac{|\alpha|}{a} \right) \|\Phi'\|^2 + (\omega - a|\alpha|) \|\Phi\|^2 - \|\Phi\|_{2\mu+2}^{2\mu+2} \geq c \|\Phi\|_{H^1}^2 - \|\Phi\|_{2\mu+2}^{2\mu+2}. \quad (4.13)$$

By Sobolev type inequalities we arrive at

$$c \|\Phi\|_{2\mu+2}^2 - \|\Phi\|_{2\mu+2}^{2\mu+2} \leq 0 \quad (4.14)$$

which implies, for a non vanishing function  $\Phi$ ,

$$\|\Phi\|_{2\mu+2} \geq c. \quad (4.15)$$

Since on the Nehari manifold  $S_\omega$  and  $\tilde{S}$  coincide, we must have  $d(\omega) > 0$ .

□

For any  $\omega > 0$  define  $\alpha^*$  such that  $-N\sqrt{\omega} < \alpha^* < 0$  and

$$\int_0^1 (1-t^2)^{\frac{1}{\mu}} dt = \frac{N}{2} \int_{\frac{|\alpha^*|}{N\sqrt{\omega}}}^1 (1-t^2)^{\frac{1}{\mu}} dt. \quad (4.16)$$

Notice that  $\alpha^*$  is uniquely defined since the r.h.s. of (4.16) is a decreasing function of  $|\alpha^*|$  whose range includes the value  $\int_0^1 (1-t^2)^{\frac{1}{\mu}} dt$ .

**Proposition 4.1.** *Let  $-N\sqrt{\omega} < \alpha < \alpha^*$ . Then*

$$d(\omega) < d^0(\omega). \quad (4.17)$$

**Proof**

In order to prove (4.17), it is sufficient to exhibit a trial function  $\tilde{\Psi} \in \mathcal{E}$  such that  $I_\omega[\tilde{\Psi}] = 0$  and  $S_\omega[\tilde{\Psi}] < d^0(\omega)$ . Let  $a$  be defined as

$$a = \frac{1}{\mu\sqrt{\omega}} \operatorname{arctanh} \left( \frac{|\alpha|}{N\sqrt{\omega}} \right).$$

Now we consider the symmetric trial function  $\tilde{\Psi}$  given by:

$$(\tilde{\Psi})_i(x) = \phi_s(x+a) \quad i = 1 \dots N \quad (4.18)$$

where  $\phi_s$  is defined by (B.1). By construction  $\tilde{\Psi} \in \mathcal{E}$ . Moreover it is straightforward to check that  $\tilde{\Psi} \in \mathcal{D}(H)$  and

$$H\tilde{\Psi} - |\tilde{\Psi}|^{2\mu}\tilde{\Psi} = -\omega\tilde{\Psi}. \quad (4.19)$$

Multiplying both sides of (4.19) and integrating by parts, one checks that  $I_\omega[\tilde{\Psi}] = 0$  that is  $\tilde{\Psi}$  satisfies the constraint. Therefore it is sufficient to evaluate the reduced action using (B.5). One has

$$S_\omega[\tilde{\Psi}] = \frac{N}{2}(\mu+1)^{\frac{1}{\mu}}\omega^{\frac{1}{\mu}+\frac{1}{2}} \int_{\frac{|\alpha|}{N\sqrt{\omega}}}^1 (1-t^2)^{\frac{1}{\mu}} dt \quad (4.20)$$

and the condition  $S_\omega[\tilde{\Psi}] < d^0(\omega)$  amounts to

$$\frac{N}{2} \int_{\frac{|\alpha|}{N\sqrt{\omega}}}^1 (1-t^2)^{\frac{1}{\mu}} dt < \int_0^1 (1-t^2)^{\frac{1}{\mu}} dt$$

which holds true by the hypothesis  $-N\sqrt{\omega} < \alpha < \alpha^*$  since

$$\frac{N}{2} \int_{\frac{|\alpha|}{N\sqrt{\omega}}}^1 (1-t^2)^{\frac{1}{\mu}} dt < \frac{N}{2} \int_{\frac{|\alpha^*|}{N\sqrt{\omega}}}^1 (1-t^2)^{\frac{1}{\mu}} dt = \int_0^1 (1-t^2)^{\frac{1}{\mu}} dt.$$

□

Now we can finally prove Theorem 1, as stated in the introduction.

**Proof**

Let  $\{\Psi_n\}$  be a minimizing sequence, we prove that there exists a subsequence weakly convergent in  $H^1$ . First notice that  $\|\Psi_n\|_{2\mu+2}$  is obviously bounded (see Lemma 4.1). Recall that for  $\Phi \in \mathcal{E}$

$$\|\Phi'\|^2 + \alpha|\phi_1(0)|^2 \geq \frac{\alpha^2}{N^2} \|\Phi\|^2. \quad (4.21)$$



Using (4.21) and  $I_\omega[\Psi_n] \leq 0$  we have

$$0 \leq \left( \omega - \frac{\alpha^2}{N^2} \right) \|\Psi_n\|^2 \leq \|\Psi'_n\|^2 + \omega \|\Psi_n\|^2 + \alpha |\psi_{n,1}(0)|^2 \leq \|\Psi_n\|_{2\mu+2}^{2\mu+2} \leq c. \quad (4.22)$$

This implies  $\|\Psi_n\| \leq c$ . Using again  $I_\omega[\Psi_n] \leq 0$  we have also

$$\|\Psi'_n\|^2 \leq \|\Psi_n\|_{2\mu+2}^{2\mu+2} - \omega \|\Psi_n\|^2 - \alpha |\psi_{n,1}(0)|^2 \leq c + c \|\psi'_{n,1}\| \|\psi_{n,1}\| \leq c + c \left( \frac{1}{\varepsilon} \|\Psi_n\|^2 + \varepsilon \|\Psi'_n\|^2 \right) \quad (4.23)$$

for any  $\varepsilon > 0$ . Taking  $\varepsilon$  sufficiently small we see that  $\|\Psi'_n\|$  is bounded and therefore also  $\|\Psi'_n\|_{H^1}$  is bounded. By Banach-Alaoglu theorem there exists a weakly convergent subsequence, which will be still denoted by  $\{\Psi_n\}$ . Let  $\Psi_\infty$  be the weak limit.

Now we prove that  $\Psi_\infty \neq 0$ . To this aim we preliminarily show that  $\Psi_\infty \in \mathcal{E}$ ,  $\Psi_n(0) \rightarrow \Psi_\infty(0)$  and that  $I_\omega[\Psi_n] \rightarrow 0$ . Let  $\Lambda^j : \mathcal{G} \rightarrow \mathbb{R}$  be a function on the graph defined in the following way:  $\lambda_j(y) = e^{-y}$  and  $\lambda_i(y) = 0$  for  $i \neq j$ . Then by weak convergence and integration by parts we have

$$\psi_{j,n}(0) = (\Lambda^j, \Psi_n)_{H^1} \rightarrow (\Lambda^j, \Psi_\infty)_{H^1} = \Psi_{j,\infty}(0). \quad (4.24)$$

Since  $\psi_{j,n}(0)$  does not depend on  $j$ , the first two claims are proved. We prove the last claim by contradiction. Assume that

$$I_\omega[\Psi_n] \rightarrow 0 \quad (4.25)$$

is false, then there exists a subsequence, still denoted by  $\{\Psi_n\}$ , such that

$$\lim_{n \rightarrow \infty} I_\omega[\Psi_n] = \gamma < 0. \quad (4.26)$$

Let  $\beta_n$  be defined according to (4.5) then

$$\lim_{n \rightarrow \infty} \beta_n = \lim_{n \rightarrow \infty} \left( 1 + \frac{I_\omega[\Psi_n]}{\|\Psi_n\|_{2\mu+2}^{2\mu+2}} \right)^{\frac{1}{2\mu}} = \left( 1 + \frac{\gamma\mu}{2(\mu+1)d(\omega)} \right)^{\frac{1}{2\mu}} < 1 \quad (4.27)$$

therefore

$$\lim_{n \rightarrow \infty} \tilde{S}[\beta_n \Psi_n] = \lim_{n \rightarrow \infty} \beta_n^{2\mu+2} \tilde{S}[\Psi_n] < d(\omega) \quad (4.28)$$

and  $I_\omega[\beta_n \Psi_n] = 0$  but this contradicts the assumption that  $\Psi_n$  is a minimizing sequence. Hence  $I_\omega[\Psi_n] \rightarrow 0$ .

We proceed again by contradiction to prove that  $\Psi_\infty \neq 0$ . Assume that  $\Psi_\infty = 0$  and define

$$\rho_n = \frac{[\|\Psi'_n\|^2 + \omega \|\Psi_n\|^2]^{\frac{1}{2\mu}}}{\|\Psi_n\|_{2\mu+2}^{1+\frac{1}{\mu}}}.$$

Using (4.24), (4.26) and the contradiction hypothesis, one has

$$\lim_{n \rightarrow \infty} \rho_n = \lim_{n \rightarrow \infty} \left( 1 + \frac{I_\omega[\Psi_n] - \alpha |\psi_{1,n}(0)|^2}{\|\Psi_n\|_{2\mu+2}^{2\mu+2}} \right)^{\frac{1}{2\mu}} = 1. \quad (4.29)$$

Therefore

$$\lim_{n \rightarrow \infty} \tilde{S}[\rho_n \Psi_n] = \lim_{n \rightarrow \infty} \rho_n^{2\mu+2} \tilde{S}[\Psi_n] = d(\omega). \quad (4.30)$$

On the other hand, by direct computation one has

$$I_\omega^0[\rho_n \Psi_n] = 0. \quad (4.31)$$

Therefore, by Proposition 4.1 and Theorem 3

$$d(\omega) < S_\infty(\omega) \leq \tilde{S}[\rho_n \Psi_n]. \quad (4.32)$$

Passing to the limit, one obtains

$$d(\omega) < S_\infty(\omega) \leq d(\omega) \quad (4.33)$$

therefore the hypothesis  $\Psi_\infty = 0$  can not hold.

Now we shall prove that  $I_\omega[\Psi_\infty] \leq 0$ . We recall, see [12], Brezis and Lieb's lemma: if  $f_n$  converges weakly to  $f_\infty$  in  $L^p$ ,  $1 < p < \infty$ , then

$$\|f_n\|_p^p - \|f_n - f_\infty\|_p^p - \|f_\infty\|_p^p \rightarrow 0. \quad (4.34)$$

In our case, this implies that

$$\tilde{S}[\Psi_n] - \tilde{S}[\Psi_n - \Psi_\infty] - \tilde{S}[\Psi_\infty] \rightarrow 0 \quad (4.35)$$

and, applying (4.34) both to  $\Psi$  and  $\Psi'$ , that

$$I_\omega[\Psi_n] - I_\omega[\Psi_n - \Psi_\infty] - I_\omega[\Psi_\infty] \rightarrow 0. \quad (4.36)$$

Suppose that  $I_\omega[\Psi_\infty] > 0$ . Then, by (4.25) and (4.36),

$$\lim_{n \rightarrow \infty} I_\omega[\Psi_n - \Psi_\infty] = \lim_{n \rightarrow \infty} I_\omega[\Psi_n] - I_\omega[\Psi_\infty] = -I_\omega[\Psi_\infty] < 0. \quad (4.37)$$

Choose  $\bar{n}$  such that  $I_\omega[\Psi_n - \Psi_\infty] < 0$  for  $n > \bar{n}$ . Then by definition of  $d(\omega)$  we have

$$d(\omega) \leq \tilde{S}[\Psi_n - \Psi_\infty], \quad n > \bar{n}. \quad (4.38)$$

On the other hand, since  $\Psi_\infty \neq 0$ , by (4.35) one has

$$\lim_{n \rightarrow \infty} \tilde{S}[\Psi_n - \Psi_\infty] = \lim_{n \rightarrow \infty} \tilde{S}[\Psi_n] - \tilde{S}[\Psi_\infty] = d(\omega) - \tilde{S}[\Psi_\infty] < d(\omega) \quad (4.39)$$

and this contradicts (4.38); so it must be  $I_\omega[\Psi_\infty] \leq 0$ .

By definition  $d(\omega) \leq \tilde{S}[\Psi_\infty]$ . On the other hand, by the lower semicontinuity of the norm under weak convergence we have

$$\tilde{S}[\Psi_\infty] = \frac{\mu}{2(\mu+1)} \|\Psi_\infty\|_{2\mu+2}^{2\mu+2} \leq \lim_{n \rightarrow \infty} \frac{\mu}{2(\mu+1)} \|\Psi_n\|_{2\mu+2}^{2\mu+2} = d(\omega) \quad (4.40)$$

which implies

$$\tilde{S}[\Psi_\infty] = d(\omega) \quad (4.41)$$

and so  $\Psi_\infty$  is an absolute minimum of  $S_\omega$  constrained to Nehari manifold.

□

## 5. STATIONARY STATES

In this section we explicitly compute the stationary states of  $S_\omega$  and of  $S_\omega^0$  and identify the minimum of the action. Some of the results in this and subsequent section were announced in [4]. We denote by  $[s]$  the integer part of  $s$ .

**Theorem 4** (Stationary states of  $S_\omega$ ).

Let  $\alpha < 0$  and  $\omega > \frac{\alpha^2}{N^2}$ ; then  $S_\omega$  has  $[(N-1)/2] + 1$  critical points  $\Psi_{\omega,j}$ , with  $j = 0, \dots, [\frac{N-1}{2}]$ , given, up to permutations of the edges, by:

$$(\Psi_{\omega,j})_i(x) = \begin{cases} \phi_s(x - a_j) & i = 1, \dots, j \\ \phi_s(x + a_j) & i = j + 1, \dots, N \end{cases} \quad (5.1)$$

$$a_j = \frac{1}{\mu\sqrt{\omega}} \operatorname{arctanh} \left( \frac{\alpha}{(2j - N)\sqrt{\omega}} \right). \quad (5.2)$$

Moreover, for  $-N\sqrt{\omega} < \alpha < \alpha^*$  the function  $\Psi_{\omega,0}$  is the ground state.

### Proof

A regularity argument shows that a constrained critical point of the action  $S_\omega$  is in fact an element of the domain of the operator  $H$ . We sketch the standard proof. Any such non vanishing critical point  $\Psi$  satisfies  $S'(\Psi) = 0$ , i.e.

$$S'_\omega(\Psi)\Phi = 0, \quad \forall \Phi \in \mathcal{E}. \quad (5.3)$$

Applying (5.3) first to  $\Phi$ , then to  $\Xi = -i\Phi$ , and summing the two expressions, we find

$$B(\Psi, \Phi) - \frac{1}{2\mu + 2}(|\Psi|^{2\mu}\Psi, \Phi) + \omega(\Psi, \Phi) = 0, \quad (5.4)$$

where  $B$  is the bilinear form associated to the quadratic form  $E^{lin}$ . So, from (5.4) the following estimate holds

$$|B(\Psi, \Phi)| \leq C_\Psi \|\Phi\|, \quad \forall \Phi \in \mathcal{E}. \quad (5.5)$$

Notice that, choosing  $\Phi$  among the functions vanishing in a neighborhood of zero, we conclude from (5.5) Riesz theorem and definition of weak derivative that every  $\psi_i \in H^2(\mathbb{R}^+)$ . Thus, for a generic  $\Phi \in \mathcal{E}$  an integration by parts gives

$$2 B(\Psi, \Phi) = - \sum_{i=1}^N (\psi_i'', \phi_i)_{L^2(\mathbb{R}^+)} - \phi_1(0) \left( \alpha \psi_1(0) - \sum_{i=1}^N \psi_i'(0) \right). \quad (5.6)$$

So, from (5.5) and (5.6), we conclude that  $\Psi$  belongs to the domain  $\mathcal{D}(H)$ . Moreover, the function  $H\Psi - \frac{1}{2\mu+2}|\Psi|^{2\mu}\Psi + \omega\Psi$  belongs to  $L^2(\mathbb{R}^+)$ .

Therefore  $S'_\omega[\Psi] = 0$  is equivalent to the following equation

$$H\Psi_\omega - |\Psi_\omega|^{2\mu}\Psi_\omega = -\omega\Psi_\omega \quad \omega > 0. \quad (5.7)$$

Notice that  $H$  acts locally as the Laplacian, thus on every edge we must seek  $L^2(\mathbb{R}^+)$ -solutions to the equation

$$-\phi'' - |\phi|^{2\mu}\phi = -\omega\phi \quad \omega > 0.$$

The most general  $L^2(\mathbb{R}^+)$ -solution is  $\phi(x) = \sigma \phi_s(x - y) = \sigma [(\mu + 1)\omega]^{\frac{1}{2\mu}} \operatorname{sech}^{\frac{1}{\mu}}(\mu\sqrt{\omega}(x - y))$  where  $\sigma \in \mathbb{C}$ ,  $|\sigma| = 1$  and  $y \in \mathbb{R}$ . Therefore the components  $(\Psi_\omega)_i$  of a critical point  $\Psi_\omega$  are given by

$$(\Psi_\omega)_i(x) = \sigma_i \phi(x - y_i).$$

In order to have a solution of (5.7) it is sufficient to impose boundary conditions (1.2) such that  $\Psi_\omega \in \mathcal{D}(H)$ . The continuity condition in (1.2) implies  $\sigma_1 = \dots = \sigma_N$  and  $y_i = \varepsilon_i a$  with  $\varepsilon_i = \pm 1$  and  $a > 0$ . We can omit the dependence on  $\sigma$  without losing generality. Referring to the bell shape of the function  $\phi_s$ , we say that in the  $i$ -th edge: there is a *bump* if  $y_i > 0$ , that is, if  $\varepsilon_i = +1$ ; there is a *tail* if  $y_i < 0$ , that is, if  $\varepsilon_i = -1$ . Now we determine  $\varepsilon_i$  and  $y_i$ . The second boundary condition in (1.2) rewrites as

$$\tanh(\mu\sqrt{\omega}a) \sum_{i=1}^N \varepsilon_i = \frac{\alpha}{\sqrt{\omega}}. \quad (5.8)$$

Equation (5.8) gives as a first constraint that  $\sum_{i=1}^N \varepsilon_i$  must have the same sign of  $\alpha$ . That is the critical point must have more tails than bumps. For every such a configuration, or equivalently a choice of the set  $\{\varepsilon_i\}$ , condition (5.8) fixes uniquely  $a$ . We choose to index the solutions by the number  $j$  of bumps. Correspondingly one obtains a unique solution to (5.8) which we call  $a_j$ . In this way we arrive at (5.1) and (5.2). For instance, if  $N = 3$  then there are two stationary states, a three-tail state and a two-tail/one-bump state. They are shown in figure 1.

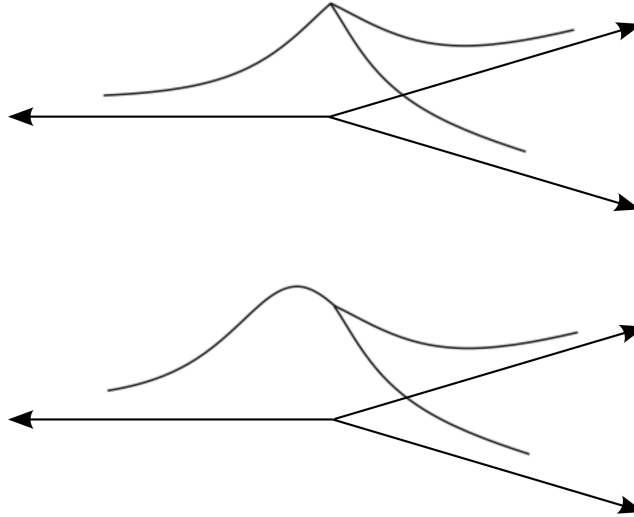


FIGURE 1. Stationary states for  $N = 3$ ,  $\alpha < 0$ .

To summarize, solutions to (5.7) are given by  $\Psi_{\omega,j}$  with  $j = 0, \dots, [(N-1)/2]$ .

Notice that (5.8) admits solutions iff the lower bound  $\frac{\alpha^2}{N^2} < \omega$  holds true. We can explain this fact for  $\alpha < 0$  noticing that (5.7), for small  $\Psi$  and neglecting nonlinearity, is the eigenvalue equation for the linear part of the Hamiltonian corresponding to energy  $E = -\omega$ ; taking into account the known fact that the linear graph Hamiltonian  $H$  has the ground state energy  $-\frac{\alpha^2}{N^2}$ , the lower bound means

that the nonlinear standing waves bifurcate from the vanishing wavefunction at the ground state energy of the linear problem.

Now we prove that  $\Psi_{\omega,0}$  is the ground state. Notice also that  $\Psi_{\omega,0}$  is uniquely defined since it is invariant under permutations of the edges. We know that for  $-N\sqrt{\omega} < \alpha < \alpha^*$  a minimum of  $S_\omega$  exists and therefore it is a critical point. It is sufficient to prove that  $S_\omega[\Psi_{\omega,0}] < S_\omega[\Psi_{\omega,j}]$  for  $j \neq 0$ . In fact we prove a stronger statement, that is, if  $0 \leq j \leq [(N-1)/2] - 1$  then

$$S_\omega[\Psi_{\omega,j}] < S_\omega[\Psi_{\omega,j+1}]. \quad (5.9)$$

Using (B.5), equation (5.9) is equivalent to

$$\begin{aligned} j \int_{-\frac{|\alpha|}{(N-2j)\sqrt{\omega}}}^1 (1-t^2)^{\frac{1}{\mu}} dt + (N-j) \int_{\frac{|\alpha|}{(N-2j)\sqrt{\omega}}}^1 (1-t^2)^{\frac{1}{\mu}} dt < \\ (j+1) \int_{-\frac{|\alpha|}{(N-2j-2)\sqrt{\omega}}}^1 (1-t^2)^{\frac{1}{\mu}} dt + (N-j-1) \int_{\frac{|\alpha|}{(N-2j-2)\sqrt{\omega}}}^1 (1-t^2)^{\frac{1}{\mu}} dt. \end{aligned} \quad (5.10)$$

Let us define the constant  $C$

$$C = \int_0^1 (1-t^2)^{\frac{1}{\mu}} dt.$$

It is convenient to rewrite the l.h.s. of (5.10) as

$$\begin{aligned} j \int_{-\frac{|\alpha|}{(N-2j)\sqrt{\omega}}}^1 (1-t^2)^{\frac{1}{\mu}} dt + (N-j) \int_{\frac{|\alpha|}{(N-2j)\sqrt{\omega}}}^1 (1-t^2)^{\frac{1}{\mu}} dt = 2jC + (N-2j) \int_{\frac{|\alpha|}{(N-2j)\sqrt{\omega}}}^1 (1-t^2)^{\frac{1}{\mu}} dt = \\ NC - (N-2j) \int_0^{\frac{|\alpha|}{(N-2j)\sqrt{\omega}}} (1-t^2)^{\frac{1}{\mu}} dt. \end{aligned} \quad (5.11)$$

Repeating the same manipulations for the r.h.s., we see that (5.11) is equivalent to

$$(N-2j-2) \int_0^{\frac{|\alpha|}{(N-2j-2)\sqrt{\omega}}} (1-t^2)^{\frac{1}{\mu}} dt < (N-2j) \int_0^{\frac{|\alpha|}{(N-2j)\sqrt{\omega}}} (1-t^2)^{\frac{1}{\mu}} dt. \quad (5.12)$$

With a straightforward change of variables the last inequality becomes

$$\int_0^{\frac{|\alpha|}{\sqrt{\omega}}} \left[ 1 - \left( \frac{t}{N-2j-2} \right)^2 \right]^{\frac{1}{\mu}} dt < \int_0^{\frac{|\alpha|}{\sqrt{\omega}}} \left[ 1 - \left( \frac{t}{N-2j} \right)^2 \right]^{\frac{1}{\mu}} dt \quad (5.13)$$

which is manifestly true.

□

**Remark 5.1.** For  $N > 2$  and  $j > 0$  there exist excited states, but only for parameters  $\omega > \frac{\alpha^2}{(N-2j)^2}$ . So the picture is that for fixed  $\alpha$  and increasing  $\omega$  firstly the branch of ground state is born at  $\omega > \frac{\alpha^2}{N^2}$  and then for sufficiently high  $\omega$  the branches of higher excited states appear.

**Remark 5.2.** *Even if in the present section we considered the case  $\alpha \leq 0$ , notice that the analysis of the previous theorem can be repeated also for  $\alpha > 0$  and one would find that the critical points are given again by (5.1) and (5.2) with  $j = [N/2 + 1], \dots, N$ . This means that for a repulsive  $\delta$  interaction at the vertex the stationary states have more bumps than tails.*

We end this section with the characterization of stationary points of  $S_\omega^0$

**Theorem 5** (Critical Points of  $S_\omega^0$ ). *Let  $\omega > 0$ . If  $N$  is odd, then there is a unique critical point of  $S_\omega^0$  given by*

$$(\Psi_\omega^0)_i = \phi_s \quad i = 1, \dots, N \quad (5.14)$$

*If  $N$  is even then  $S_\omega^0$  has a one parameter family of critical points given by:*

$$(\Psi_\omega^{0,a})_i(x) = \begin{cases} \phi_s(x-a) & i = 1, \dots, N/2 \\ \phi_s(x+a) & i = N/2 + 1, \dots, N \end{cases} \quad a \in \mathbb{R}^+ \quad (5.15)$$

### Proof

Repeating the argument in the proof of Theorem 4, we have to find the solutions of

$$\tanh(\mu\sqrt{\omega}a) \sum_{i=1}^N \varepsilon_i = 0.$$

If  $N$  is odd, then there is a unique solution given by  $a = 0$  which corresponds to the critical point (5.14), and such a solution can be described as composed by  $N$  half solitons glued at the vertex. On the contrary, if  $N$  is even then there are infinite solutions:  $a \in \mathbb{R}^+$ ,  $\varepsilon_i = +1$  for  $i = 1, \dots, N/2$ ,  $\varepsilon_i = -1$  for  $N/2 + 1, \dots, N$  gives a solution to (5.15) which corresponds to  $\Psi_\omega^{0,a}$ .

□

**Remark 5.3.** *If  $N$  is even, then the graph can be considered as a set of  $N/2$  copies of the real line. With a Kirchhoff boundary condition, one has continuity and derivability of the wavefunction at the vertex, and the above solutions  $\Psi_\omega^{0,a}$  can be interpreted as  $N/2$  identical solitary waves on each real line translated by a quantity  $a$ .*

**Remark 5.4.** *In the case  $N = 3$  and for a cubic nonlinearity it has been proved in [3] that the energy  $E$  at constant mass  $M$  is not minimized on  $\Psi_\omega^0$ , which turns out to be a saddle point. In fact, the constrained energy is bounded from below but it has not an absolute minimum. We conjecture that the same phenomenon happens here for the action.*

## 6. STABILITY OF GROUND STATES

In Section 4 we showed the existence of a profile  $\Psi_{\omega,0}$  (denoted there by  $\Psi_\infty$ ) which minimizes the action  $S_\omega$  for the star graph with attractive delta boundary conditions at the vertex. This minimizer is the ground state of the problem if the strength  $\alpha$  of the point interaction at the vertex is sufficiently large. In Section 5 we provided the explicit expression of stationary states  $\Psi_{\omega,j}$ , and in particular of the ground state  $\Psi_{\omega,0}$ . In correspondence to the ground state (and to every stationary state) one has a standing wave of the form  $\Psi_\omega e^{i\omega t}$  which solves the NLS on the graph. In this section we study the stability of such a standing wave. Being a time-dependent solution and not an equilibrium

point of the autonomous equation (1.4), stability has to be intended as *orbital stability*. This means Lyapunov stability up to symmetries of the equation, which in this case are related to the gauge  $U(1)$  invariance of the Hamiltonian of the problem. To be precise, we recall that the orbit of  $\Psi_\omega$  is defined as  $\mathcal{O}(\Psi_\omega) = \{e^{i\theta}\Psi_\omega, \theta \in \mathbb{R}\}$ .

The state  $\Psi_\omega$  is orbitally stable if for every  $\epsilon > 0$  there exists  $\delta > 0$  such that

$$d(\Psi(0), \mathcal{O}(\Psi_\omega)) < \delta \quad \Rightarrow \quad d(\Psi(t), \mathcal{O}(\Psi_\omega)) < \epsilon \quad \forall t > 0$$

where  $\Psi(t)$  is the solution to (2.1) with initial data  $\Psi_0$ ,

$$d(\Psi, \mathcal{O}(\Psi_\omega)) = \inf_{\Phi \in \mathcal{O}(\Psi_\omega)} \|\Psi - \Phi\|_{\mathcal{E}},$$

and the norm  $\|\cdot\|_{\mathcal{E}}$  is the energy norm, given in our case by  $H^1$  norm in  $\mathcal{E}$ .

A stationary state is unstable if it is not stable.

Orbital stability of solitary (not necessarily standing) solutions to nonlinear Schrödinger equations is a well developed subject, studied in several classical papers. Two main techniques have been developed to establish orbital stability of solitary waves: the method of Cazenave and Lions based on Concentration Compactness ([14, 15]), and the method of constrained linearization pioneered by Benjamin in the case of KdV equation and studied more sistematically by Weinstein and Grillakis-Shatah-Strauss [35, 36, 19, 20]. In a forthcoming paper we will treat minimization of energy at constant mass through a suitable adaptation of concentration compactness method to the case of star graphs ([1]), while here we refer to the Weinstein and Grillakis-Shatah-Strauss method which is especially suited for treating stability of equilibria of Hamiltonian systems with symmetry. Some preparation is needed to cast our problem in this framework. As in the scalar case, the NLS on a graph turns out to be a Hamiltonian system on the real Hilbert space of the couples of real and imaginary part of the wavefunction. We pose  $\Psi = U + iV \equiv (U, V)$ , where  $U = (u_1, \dots, u_N)^T$  and  $V = (v_1, \dots, v_N)^T$ . So we identify  $L^2(\mathcal{G}) = L^2(\mathcal{G}, \mathbb{C})$  with  $L^2(\mathcal{G}, \mathbb{R}) \oplus L^2(\mathcal{G}, \mathbb{R}) := L^2_{\mathbb{R}}(\mathcal{G})$ . Analogously one can define the spaces  $L^p_{\mathbb{R}}(\mathcal{G})$ . Correspondingly,  $L^2(\mathcal{G})$  can be given the structure of a real Hilbert space taking as its scalar product the real part of the usual complex one:

$$((U_1, V_1)^T, (U_2, V_2)^T)_{L^2_{\mathbb{R}}(\mathcal{G})} = \text{Re}(\Psi_1, \Psi_2)_{L^2(\mathcal{G})}. \quad (6.1)$$

Furthermore,  $L^2(\mathcal{G})$  is also a symplectic manifold when endowed with the symplectic form (coinciding with the imaginary part of the complex scalar product)

$$\Omega((U_1, V_1), (U_2, V_2)) = \text{Im}(\Psi_1, \Psi_2)_{L^2(\mathcal{G})} = \sum_{i=1}^N \int_{\mathbb{R}^+} ((v_2)_i (u_1)_i - (v_1)_i (u_2)_i) dx.$$

The same symplectic structure is inherited by the energy space  $\mathcal{E}$ . Moreover, multiplication by the imaginary unit  $i$  is equivalent to acting by the matrix  $-\mathcal{J} \in \text{Mat}(\mathbb{R}, 2N \times 2N)$ , where

$$\mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

and the blocks 0 and  $I$  are the zero and unit matrices in  $\text{Mat}(\mathbb{R}, N \times N)$ .

Note that if  $\Psi \in \mathcal{D}(H)$ , then the real vectors  $U$  and  $V$  satisfy the same boundary conditions as  $\Psi$ ; we will say, with a slight abuse, that they belong to  $\mathcal{D}(H)$ . With these premises, the nonlinear

Schrödinger equation for  $\Psi$  is equivalent to the canonical system

$$\frac{d}{dt} \begin{pmatrix} U \\ V \end{pmatrix} = \mathcal{J}E'[U, V]. \quad (6.2)$$

where the Hamiltonian  $E$  becomes

$$E(U, V) = \frac{1}{2} \|(U', V')\|_{L^2_{\mathbb{R}}(\mathcal{G})}^2 - \frac{1}{2\mu + 2} \|(U, V)\|_{L^{2\mu+2}_{\mathbb{R}}(\mathcal{G})}^{2\mu+2} + \frac{\alpha}{2} (|u_1(0)|^2 + |v_1(0)|^2) \quad (6.3)$$

or explicitly

$$\begin{aligned} & \frac{1}{2} \sum_{k=1}^N \left[ \int_0^{+\infty} (|u'_k|^2 + |v'_k|^2) dx \right] + \frac{\alpha}{2} (|u_1(0)|^2 + |v_1(0)|^2) \\ & - \frac{1}{2\mu + 2} \sum_{k=1}^N \left[ \int_0^{\infty} (|u_k|^2 + |v_k|^2)^{\mu+1} dx \right] \equiv E[u, v], \end{aligned} \quad (6.4)$$

and the derivative  $E'$  is given by

$$E'((U, V))[(H, Z)] = \frac{d}{d\epsilon} \{E((U, V) + \epsilon(H, Z))\}_{\epsilon=0}.$$

Linearization of the Hamiltonian system (6.2) around the stationary state is achieved by substituting

$$(\Psi_t)_j = ((\Psi_{\omega,0})_j + h_j + iz_j)e^{i\omega t} \quad (6.5)$$

and neglecting higher order terms than linear in (6.2). The real vector functions  $H$  and  $Z$  satisfy

$$\frac{d}{dt} \begin{pmatrix} H \\ Z \end{pmatrix} = \mathcal{J}\mathcal{L} \begin{pmatrix} H \\ Z \end{pmatrix},$$

where  $\mathcal{L}$  is the unique s.a. operator associated to the symmetric and lower bounded quadratic form  $E''(\Psi_{\omega,0})$ , i.e. the second derivative of the energy at the ground state. Indeed, the second derivative is defined through the formula

$$\mathcal{L}((H_1, Z_1), (H_2, Z_2)) = E''(\Psi_{\omega,0})((H_1, Z_1), (H_2, Z_2)) = \frac{\partial^2}{\partial \epsilon \partial \lambda} \{E(\Psi_{\omega} + \epsilon(H_1, Z_1) + \lambda(H_2, Z_2))\}_{\epsilon=0, \lambda=0}.$$

An easy computation shows that  $\mathcal{L} = \text{diag}(\mathcal{L}_-, \mathcal{L}_+)$  and the matrix operators  $\mathcal{L}_-$  and  $\mathcal{L}_+$  are given by (here the summation convention is used)

$$\begin{aligned} (\mathcal{L}_+)_{i,k} &= \left( -\frac{d^2}{dx^2} + \omega - |(\Psi_{\omega,0})_k|^{2\mu} \right) \delta_{i,k} \\ (\mathcal{L}_-)_{i,k} &= \left( -\frac{d^2}{dx^2} + \omega - (2\mu + 1)|(\Psi_{\omega,0})_k|^{2\mu} \right) \delta_{i,k}. \end{aligned}$$

The operators  $\mathcal{L}_-$  and  $\mathcal{L}_+$  act on the real vector functions  $H$  and  $Z$  belonging to  $\mathcal{D}(H) = \mathcal{D}(\mathcal{L}_{\pm})$ . Notice that to simplify notation from here on we suppress the dependence of operators  $\mathcal{L}_{\pm}$  on the ground state  $\Psi_{\omega,0}$ .

Precise conditions to have orbital stability (and instability) for general Hamiltonian systems and in particular for systems of NLS equations, are given in the already quoted papers of Weinstein and Grillakis-Shatah-Strauss. They can be reduced to the validity of three conditions, called Assumptions I, II and III in [19] and [20] and the verification of a further convexity condition on the function



$d(\omega) = S_\omega(\Psi_\omega)$  introduced in Section 4 and called in the physical literature the Vakitov-Kolokolov condition.

Assumption I is the well posedness, proved in Section 2. Assumption II is the existence of a regular branch of standing solutions of the stationary equation, proved for our model in Section 4 and 5, where the regular family of standing waves  $(\frac{\alpha^2}{N^2}, +\infty) \ni \omega \mapsto \Psi_{\omega,0}$  is explicitly constructed. Assumption III concerns spectral properties of linearization  $E''(\Psi_{\omega,0}) = (\mathcal{L}_-, \mathcal{L}_+)$  around the ground state. The spectral conditions are stated and proved in the following proposition.

**Proposition 6.1.** *The operators  $\mathcal{L}_-$  and  $\mathcal{L}_+$  are selfadjoint. Moreover:*

*$i_1)$   $\ker \mathcal{L}_+ = \{\Psi_{\omega,0}\}$  and the rest of the spectrum is positive;*

*$i_2)$   $\ker \mathcal{L}_- = \{0\}$ ;*

*$i_3)$   $n(\mathcal{L}_-) = 1$ , where  $n(A)$  is the number of negative eigenvalues of the operator  $A$ , i.e. its Morse index.*

### Proof

We begin to remark that operators  $\mathcal{L}_-$  and  $\mathcal{L}_+$  are selfadjoint on  $\mathcal{D}(\mathcal{L}_\pm) = \mathcal{D}(H)$ , due to the fact that the components of the ground state  $(\Psi_{\omega,0})_k$  are continuous and strongly decaying at infinity, and as such they constitute in the matrix operators  $\mathcal{L}_\pm$  a relatively compact perturbation of  $H + \omega$ . For the same reason, by Weyl's theorem the absolutely continuous spectrum of  $\mathcal{L}_-$  and  $\mathcal{L}_+$  coincides with the essential spectrum of  $H + \omega$ , i.e.  $[\omega, +\infty)$ , and the discrete spectrum is composed at most of a finite number of eigenvalues. Let us consider the kernel of  $\mathcal{L}_+$ . This surely contains  $\Psi_{\omega,0}$ . Indeed, the equation  $\mathcal{L}_+ \Psi_{\omega,0} = 0$  coincides with the stationary equation satisfied by  $\Psi_{\omega,0}$ . Let us show that there are not other elements in the kernel. An integration by parts allows to rewrite the quadratic form of  $\mathcal{L}_+$ , for any element  $V \in \mathcal{D}(\mathcal{L}_+)$ , as follows

$$\begin{aligned} (\mathcal{L}_+ V, V)_{L^2_{\mathbb{R}}(\mathcal{G})} &= \sum_{k=1}^N \int_0^{+\infty} ((\Psi_{\omega,0})_k)^2 \left| \frac{d}{dx} \left( \frac{v_k}{(\Psi_{\omega,0})_k} \right) \right|^2 dx \\ &\quad + \sum_{k=1}^N \left( v_k(0) v'_k(0) - |v_k(0)|^2 \frac{(\Psi'_{\omega,0})_k(0)}{(\Psi_{\omega,0})_k(0)} \right) \end{aligned} \quad (6.6)$$

and the last term is vanishing due to the  $\delta$  boundary conditions at the vertex, continuity and  $\sum_{k=1}^N v'_k(0) = \alpha v_1(0)$ .

So  $(\mathcal{L}_+ v, v)_{L^2_{\mathbb{R}}(\mathcal{G})} > 0$  for every  $v \in \mathcal{D}(\mathcal{L}_+)$  not coinciding with  $\Psi_{\omega,0}$ , which is the only eigenvector with eigenvalue 0 of the operator  $\mathcal{L}_+$ . This proves statement  $i_1$ .

Concerning statement  $i_2$ , it is sufficient to consider the equation  $\mathcal{L}_- u = 0$ . This is written, in components, as

$$-\frac{d^2}{dx^2} u_k + \omega u_k - (2\mu + 1) |(\Psi_{\omega,0})_k|^{2\mu} u_k = 0 \quad k = 1 \cdots N, \quad (6.7)$$

where  $u_1(0) = u_2(0) = \cdots = u_N(0)$  and  $\sum_{k=1}^N u'_k(0) = \alpha u_1(0)$  due to boundary conditions. The general theory of second order differential equations gives for the previous equation a solution which is a linear combination of asymptotically exponential fundamental solutions; for  $x \rightarrow +\infty$  only one of them is in  $L^2$ . Now notice that the function  $\tilde{u}_k : (0, +\infty) \mapsto \mathbb{R}$  s.t.  $\tilde{u}_k(x) = \frac{d}{dx}(\Psi_{\omega,0})_k(x)$ ,  $k = 1, \dots, N$  satisfies equation (6.7) in  $(0, +\infty)$  and it is decaying at infinity. So every solution in  $u \in L^2(0, +\infty)$  of (6.7) is a multiple of such a function:  $u_k(x) = c_k \tilde{u}_k(x)$ ,  $c_k \in \mathbb{R}$ . To conclude, a

direct calculation shows that the only real constants compatible with the boundary conditions on  $u$  is  $c_k = 0$ ,  $k = 1, \dots, N$ . This proves statement  $i_2$ .

Let us consider finally the Morse index of the  $\mathcal{L}_-$  operator. It is immediate that  $n(\mathcal{L}_-) \geq 1$ . Indeed, let us consider the quadratic form for  $\mathcal{L}_-$  evaluated on the ground state:

$$(\mathcal{L}_- \Psi_{\omega,0}, \Psi_{\omega,0})_{L^2_{\mathbb{R}}(\mathcal{G})} = (\mathcal{L}_+ \Psi_{\omega,0}, \Psi_{\omega,0})_{L^2_{\mathbb{R}}(\mathcal{G})} - 2\mu(|\Psi_{\omega,0}|^{2\mu} \Psi_{\omega,0}, \Psi_{\omega,0})_{L^2_{\mathbb{R}}(\mathcal{G})} = 0 - 2\|\Psi_{\omega,0}\|_{2\mu+2}^{2\mu+2} < 0. \quad (6.8)$$

So the s.a. linear operator  $\mathcal{L}_-$  has a negative vector, so it surely admits at least negative eigenvalue. Let us prove that it has a single negative eigenvalue only. This is a consequence of the variational properties of  $\Psi_{\omega,0}$ . In fact  $\Psi_{\omega,0}$  is a minimum point of the action  $S_\omega$  on the codimension one constraint  $I_\omega = 0$ . This minimization property entails that  $S''(\Psi_{\omega,0})$  is positive definite on the tangent space at  $\Psi_{\omega,0}$  of the constraint manifold. Being the constraint a manifold of codimension one,  $S''(\Psi_{\omega,0})$  admits at most one negative eigenvalue and the same is true for its only possibly negative diagonal component  $\mathcal{L}_-$ . See Appendix B in [16] for the detailed argument. □

The last property needed to show orbital stability of the ground state is the so called slope condition, or Vakhitov-Kolokolov condition. This coincides with the convexity of the function  $d(\omega)$ , or more explicitly it means that on the branch of stationary solutions  $\{\Psi_{\omega,0}\}$  parametrized by  $\omega$ , one has

$$d''(\omega) = \frac{d^2}{d\omega^2} S_\omega(\Psi_\omega) = \frac{d^2}{d\omega^2} (E(\Psi_\omega) + \omega M(\Psi_\omega)) = \frac{d}{d\omega} \|\Psi_{\omega,0}\|^2 > 0.$$

In fact, a direct calculation making use of the formulas in the appendix (and which is possible in this model due to the explicitly known form of  $\Psi_{\omega,0}$ ), gives

$$\frac{d}{d\omega} \|\Psi_{\omega,0}\|_2^2 = C \left[ \left( \frac{1}{\mu} - \frac{1}{2} \right) \int_{\frac{|\alpha|}{N\sqrt{\omega}}}^1 (1-t^2)^{\frac{1}{\mu}-1} dt + \frac{|\alpha|}{2N\sqrt{\omega}} \left( 1 - \frac{|\alpha|^2}{N^2\omega} \right)^{\frac{1}{\mu}-1} \right] \quad (6.9)$$

with  $C = C(N, \mu, \omega) = N^{\frac{(\mu+1)\frac{1}{\mu}}{\mu}} \omega^{\frac{1}{\mu}-\frac{3}{2}}$ .

Now, the r.h.s of (6.9) is positive thanks to the lower bound on  $\omega > \frac{\alpha^2}{N^2}$ . The Vakhitov-Kolokolov condition with the spectral properties proved in proposition 6.1, thanks to the Weinstein and Grillakis-Shatah-Strauss theory constitute the proof of Theorem 2 stated in the introduction.

**Remark 6.1.** *Note the following facts.*

*The theorem gives orbital stability of the ground state also for the critical nonlinearity  $\mu = 2$ .*

*From formula (6.9) it follows that for supercritical nonlinearities  $\mu > 2$  there exists  $\omega^*$  such that  $\Psi_{\omega,0}$  is orbitally stable for  $\omega \in (\frac{\alpha^2}{N^2}, \omega^*)$ . In [19] it is shown that if Assumptions I, II, III are satisfied and  $d''(\omega) < 0$ , then the standing wave corresponding to  $\omega$  is orbitally unstable. Again from formula (6.9) we see that for  $\omega > \omega^*$  the ground state  $\Psi_{\omega,0}$  is orbitally unstable. The case  $\omega = \omega^*$  where  $d''(\omega) = 0$  is undecided.*

## APPENDIX A. REARRANGEMENTS

For a given function  $\Phi : \mathcal{G} \rightarrow \mathbb{C}$  we introduce the rearranged function  $\Phi^* : \mathcal{G} \rightarrow \mathbb{R}$ . The function  $\Phi^*$  is positive, symmetric, non increasing and is constructed in a way such that it is equimeasurable w.r.t.  $\Phi$ , that is, the level sets of  $|\Phi|$  and  $\Phi^*$  have the same measure. This is sufficient to prove that all the

$L^p(\mathcal{G})$  norms are conserved by the rearrangement. The comparison of the kinetic energy of  $\Phi$  and  $\Phi^*$  is more delicate. On the real line the Pólya-Szegő inequality shows that the kinetic energy does not increase. This is no longer true for a star graph where a constant  $N/2$  appears, see Theorem 6 below.

Given  $\Phi : \mathcal{G} \rightarrow \mathbb{C}$ , we introduce  $\lambda(s)$  and  $\mu(s)$  defined by

$$\lambda(s) = |\{|\Phi| \geq s\}| \quad \mu(s) = |\{|\Phi| > s\}|.$$

Now we define the symmetric rearrangement of  $\Phi$ .

**Definition A.1.** Define  $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  as

$$g(t) = \sup\{s \mid \lambda(s) > Nt\},$$

then we put  $\Phi^* = (\phi_1^*, \dots, \phi_N^*)$  with

$$\phi_j^*(x) = g(x) \quad j = 1, \dots, N.$$

The main properties of  $\Phi^*$  are the following:

**Proposition A.1.** Let  $\Phi \in L^p(\mathcal{G})$ . The symmetric rearrangement  $\Phi^*$  is positive, symmetric and non increasing. Moreover  $\|\Phi^*\|_p = \|\Phi\|_p$ .

### Proof

By construction  $\Phi^*$  is positive and symmetric. Since  $\lambda$  is non increasing,  $\Phi^*$  is non increasing too. Now we prove the invariance of the  $L^p$  norm. First we prove that if  $0 \leq Nt \leq \mu(0)$  then

$$\mu(g(t)) \leq Nt \leq \lambda(g(t)). \quad (\text{A.1})$$

Let  $s' < g(t)$  then, by definition of  $g$ ,  $\lambda(s') > Nt$ . Since the latter estimate holds for every such  $s'$ , taking the supremum we have  $\lambda(g(t)) \geq Nt$  which is the second half of (A.1).

Now choose  $s' > Nt$ . Then  $\mu(s') \leq \lambda(s') \leq Nt$  and taking the infimum over  $s'$  the proof of (A.1) is complete.

A key property of the symmetric rearrangement is the equimisurability, that is:

$$|\{\Phi^* \geq s\}| = |\{|\Phi| \geq s\}|. \quad (\text{A.2})$$

Notice that the set  $\{|\Phi| \geq t\}$  can be rephrased as the union of disjoint sets

$$\{|\Phi| \geq t\} = \{|\phi_1| \geq t\} \cup \dots \cup \{|\phi_N| \geq t\},$$

where  $\{|\phi_j| \geq t\}$  is to be understood as a subset of the  $j$ -th edge. Therefore we have

$$|\{|\Phi| \geq t\}| = \sum_{i=1}^N |\{|\phi_i| \geq t\}|.$$

Fix  $s$  and define

$$t_0 = \sup\{t \mid g(t) \geq s\}.$$

If  $g(t) > s$  then  $Nt \leq \lambda(g(t)) \leq \lambda(s)$  by (A.1). Taking the supremum over  $t$  we get  $Nt_0 \leq \lambda(s)$ . Assume by absurd that  $Nt_0 < \lambda(s)$ . Take  $t$  such that  $Nt_0 < Nt < \lambda(s)$  then  $g(t) \geq s$  and the contradiction with the definition of  $t_0$  is reached. Since  $Nt_0 = |\{\Phi^* \geq s\}|$  equality (A.2) is proved.

By the layer cake representation (see [27]) we can prove the invariance of  $L^p$  norm under rearrangements. We start from

$$\|\Phi\|_p^p \equiv \sum_{j=1}^N \int_{\mathbb{R}^+} |\phi_j(y)|^p dy = p \int_0^{+\infty} s^{p-1} |\{|\Phi| \geq s\}| ds$$

and using (A.2) we obtain

$$\|\Phi\|_p^p = p \int_0^{+\infty} s^{p-1} |\{\Phi^* \geq s\}| ds = \|\Phi^*\|_p^p$$

which is the desired identity.

□

Notice that if  $\{|\Phi| = t\}$  has non-zero measure for some  $t$  then  $\lambda$  has a jump, while if  $|\Phi|$  has a jump then  $\lambda$  has a constant part. Moreover if  $\lambda$  has a constant part then  $g$  has a jump and if  $\lambda$  has a jump then  $g$  has a constant part.

Notice also that if  $|\Phi|$  is continuous and  $\{|\Phi| = t\}$  has zero measure for  $t$  then  $g$  is the inverse of  $\lambda$  up to scaling.

Now we turn our attention to the Pólya-Szegő inequality and prove it by elementary methods. We mainly follow [21] while some technical results are taken from [22].

From now on, we restrict ourselves to real and positive  $\Phi$ . Later we shall extend the result to the general case. First we gather some preliminary results in Lemmas A.1 and A.2. Then we establish the required estimate for a class of regular functions (Lemma A.3 and Lemma A.2). Finally we give the main theorem the proof of which relies on a careful decomposition of the kinetic energy and on a limiting argument.

Moreover, since functions in  $\mathcal{E}$  are continuous, for our purposes in the following we always assume that  $\Phi$  is continuous without losing generality.

**Lemma A.1.** *Assume that  $\Phi : \mathcal{G} \rightarrow \mathbb{R}^+$  is continuous and  $\Phi \in L^p(\mathcal{G})$  then  $\Phi^*$  is continuous and  $\Phi^* \in L^p(\mathcal{G})$ .*

### Proof

Due to proposition A.1, we have to discuss the continuity part only. Since  $\Phi$  is continuous,  $\lambda$  is strictly decreasing and may have at most a countable number of discontinuity of first kind. Therefore  $\lambda$  is locally continuous away from discontinuities and  $g$  is locally (up to an irrelevant scaling) the inverse function. Then  $g$  is locally continuous by the inverse function theorem. At the points where  $\lambda$  has a discontinuity it is easy to check, using definition A.1, that  $g$  has a constant part joining the non constant branches. So that  $g$  is globally continuous. See [21] for more details.

□

Reasoning as in [21] we get the following:

**Lemma A.2.** Let  $\Phi_n, \Phi : \mathcal{G} \rightarrow \mathbb{R}^+$  and  $\Phi_n, \Phi \in L^p(\mathcal{G})$ . Suppose that  $\|\Phi_n - \Phi\|_p \rightarrow 0$ , then

$$\begin{aligned} \mu(s) &\leq \liminf_n \lambda_n(s) \leq \limsup_n \lambda_n(s) \leq \lambda(s), \\ g(0) &\leq \liminf_n g_n(0). \end{aligned}$$

We introduce the following class of regular functions.

**Definition A.2.** Let  $\mathcal{PL}$  be the set of functions  $\Phi : \mathcal{G} \rightarrow \mathbb{R}^+$  such that:  $\Phi$  is continuous, compactly supported and, for any  $j = 1, \dots, N$ , there exists a finite number of compact intervals  $I_{j,n}$  such that  $\text{supp } \phi_j = \bigcup_n I_{j,n}$  and  $\Phi$  restricted to  $I_{j,n}$  is affine.

This class of piecewise linear functions is dense in  $\mathcal{E}$ .

**Lemma A.3.** Let  $\Phi \in \mathcal{PL}$ . There exist two open sets  $O_1$  and  $O_2$  such that  $\Phi^*$  is constant on  $O_1$  and  $\Phi^*$  is differentiable on  $O_2$  with  $|\Phi^{*'}| > 0$ . Moreover,  $\mathcal{G} \setminus (O_1 \cup O_2)$  consists of finitely many points.

### Proof

Let  $0 = a_0 < a_1 < \dots < a_m$  be the values assumed by  $\Phi$  at the boundary of all  $I_{j,n}$ . If the set  $\cup_i \{\Phi = a_i\}$  has positive measure then  $\Phi' = 0$  a.e. on such a set. In the same way  $\Phi^{*'} = 0$  a.e. on the set  $\cup_i \{\Phi^* = a_i\}$ . We define  $O_1$  to be the interior part of  $\cup_i \{\Phi^* = a_i\}$  and  $O_2 = \mathcal{G} \setminus \cup_i \{\Phi^* = a_i\}$ . By construction  $\mathcal{G} \setminus (O_1 \cup O_2)$  consists of finitely many points.

We have to show that  $\Phi^*$  is differentiable on  $O_2$  and  $|\Phi^{*'}| > 0$ . We introduce the notation  $D_i = \{a_{i-1} < \Phi < a_i\}$  and  $D_i^* = \{a_{i-1} < \Phi^* < a_i\}$ . Each set  $D_i$  is decomposed first into the components on each edge, that is,  $D_i = \cup_j D_i^j$ . Then we further decompose each  $D_i^j$  into a finite union of open intervals  $D_{i,k}^j$ ,  $k = 0, 1, \dots, n = n(i, j)$ , such that  $\Phi$  restricted to  $D_{i,k}^j$  is affine and non constant (see Figure 2). Let us fix  $s$  such that  $a_{i-1} < s < a_i$ . Then the equation  $\Phi = s$  has a solution  $y_{i,k}^j(s) \in D_{i,k}^j$  for each  $k = 1, \dots, n$ . We enumerate the sets  $D_{i,k}^j$  in  $k$  for  $i, j$  fixed in an increasing way w.r.t to the distance from the vertex such that  $y_{i,1}^j < y_{i,2}^j < \dots < y_{i,n}^j$ . We put  $n(i, j) = 0$  if  $D_i^j = \emptyset$  and no  $y_{i,k}^j$  is defined for that values of  $i$  and  $j$ .

We introduce also  $\tilde{D}_i^*$  defined as the projection of  $D_i^*$  on the first edge. Let  $y^*(s) \in \tilde{D}_i^*$  be the solution to  $g = s$ . We can express the measure of the level set  $\{\Phi > s\}$  by means of the  $y_{i,k}^j$ . One has

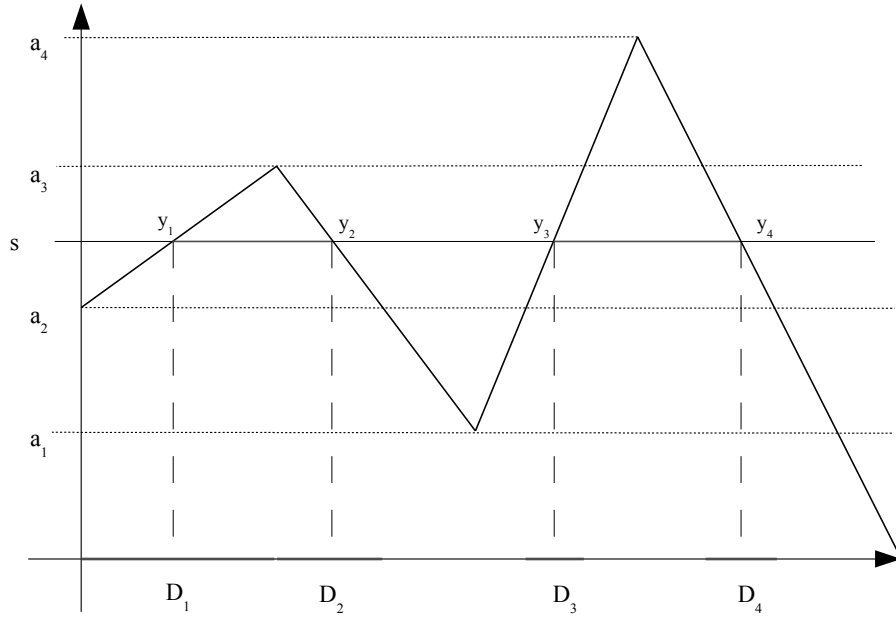
$$|\{\Phi > s\}| = \sum_{j=1}^N y_{i,n}^j - y_{i,n-1}^j + y_{i,n-2}^j - y_{i,n-3}^j \dots = \sum_{j=1}^N \sum_{k=1}^n (-1)^{n+k} y_{i,k}^j.$$

Therefore by (A.2) one has

$$|\{\Phi^* > s\}| = N |\{g > s\}| = N y^* = \sum_{j=1}^N \sum_{k=1}^n (-1)^{n+k} y_{i,k}^j. \quad (\text{A.3})$$

On each set  $D_{i,k}^j$  the derivative  $\Phi'$  does not vanish by construction which implies that the function  $y_{i,k}^j(s)$  are differentiable w.r.t  $s$  and

$$\Phi' = \left( \frac{dy_{i,k}^j}{ds} \right)^{-1} \quad \text{on } D_{i,k}^j$$

FIGURE 2. Definition and enumeration of  $D_{i,k}^j$ 

by the inverse function theorem. The function  $y^*(s)$  is differentiable by equation (A.3). It is also invertible since  $s$  is away from values of  $\Phi$  corresponding to level sets with non zero measure. Therefore for such values of  $s$  the function  $g$  is strictly decreasing. By the inverse function theorem  $y^*$  is invertible and

$$g' = \left( \frac{dy^*}{ds} \right)^{-1} \neq 0.$$

□

Let  $L$  be the Lipschitz constant of  $\Phi$ . Adapting the reasoning in [22], one can prove the following estimate:

$$\left| \frac{dy^*}{ds} \right| \geq \frac{1}{L}.$$

This estimate provides an upper bound on  $g'$ . Notice that the proof actually shows that  $\Phi^* \in \mathcal{PL}$  since it says that  $y^{*'} and therefore  $g'$  is locally constant on each  $\tilde{D}_i^*$ . If  $\Phi$  is smooth, say  $C^k$ , then the same property holds on  $O_2$  for  $\Phi^*$  by the inverse function theorem.$

**Proposition A.2.** *Let  $\Phi \in \mathcal{PL}$ . Then,*

$$\|\Phi^{*'}\| \leq \frac{N}{2} \|\Phi'\|. \quad (\text{A.4})$$

### Proof

We shall use the notation of the previous lemma. First we consider the r.h.s. of (A.4). We can

restrict the integral to the region where  $\Phi$  is not constant and change the integration variable.

$$\|\Phi'\|^2 = \sum_{i=1}^m \sum_{j=1}^N \sum_{k=1}^n \int_{D_{i,k}^j} |\Phi'|^2 = \sum_{i=1}^m \sum_{j=1}^N \sum_{k=1}^n \int_{a_{i-1}}^{a_i} \left( \frac{dy_{i,k}^j}{ds} \right)^{-2} \left| \frac{dy_{i,k}^j}{ds} \right| ds.$$

We can repeat the same operation for the l.h.s. of (A.4)

$$\|\Phi^{*'}\|^2 = N \int_{\mathbb{R}^+} |g'|^2 = N \sum_{i=1}^m \int_{a_{i-1}}^{a_i} \left( \frac{dy^{*'}}{ds} \right)^{-2} \left| \frac{dy^*}{ds} \right| ds.$$

Now the conclusion follows by using (A.3) and the convexity properties of the square function. First notice that

$$\left| \frac{dy^*}{ds} \right| = \frac{1}{N} \left| \sum_{j=1}^N \sum_{k=1}^n (-1)^{n+k} \frac{dy_{i,k}^j}{ds} \right| = \frac{1}{N} \sum_{j=1}^N \sum_{k=1}^n \left| \frac{dy_{i,k}^j}{ds} \right|.$$

The restriction of  $\Phi$  to an edge has a seesaw behavior and  $\frac{dy_{i,k}^j}{ds}$  has an alternating behavior in  $k$ . Therefore in order to prove (A.4) it is sufficient to show that

$$\frac{N^2}{4} \sum_{j=1}^N \sum_{k=1}^n \left( \frac{1}{\left| \frac{dy_{i,k}^j}{ds} \right|} \right)^2 \left| \frac{dy_{i,k}^j}{ds} \right| \geq \left( \sum_{j=1}^N \sum_{k=1}^n \frac{N}{\left| \frac{dy_{i,k}^j}{ds} \right|} \right)^2 \sum_{l=1}^N \sum_{h=1}^n \left| \frac{dy_{i,h}^l}{ds} \right|$$

which is equivalent to

$$\frac{1}{4} \sum_{j=1}^N \sum_{k=1}^n \left( \frac{1}{\left| \frac{dy_{i,k}^j}{ds} \right|} \right)^2 \frac{\left| \frac{dy_{i,k}^j}{ds} \right|}{\sum_{l=1}^N \sum_{h=1}^n \left| \frac{dy_{i,h}^l}{ds} \right|} \geq \left( \sum_{j=1}^N \sum_{k=1}^n \frac{1}{\left| \frac{dy_{i,k}^j}{ds} \right|} \right)^2. \quad (\text{A.5})$$

By the convexity of the square function, inequality (A.5) holds true if

$$\frac{1}{4} \left( \sum_{j=1}^N \sum_{k=1}^n \right)^2 \geq 1. \quad (\text{A.6})$$

Notice that  $\sum_{j=1}^N \sum_{k=1}^n$  represents the number of solutions to the equation  $\Phi = s$  for  $a_{i-1} < s < a_i$  on the whole graph. Since  $\Phi$  is continuous and compactly supported, there are always at least two solutions and then (A.6) holds true.

□

**Theorem 6** (Pólya - Szegő inequality).

Assume that  $\Phi \in \mathcal{E}$  then  $\Phi^* \in \mathcal{E}$  and

$$\|\Phi^{*'}\| \leq \frac{N}{2} \|\Phi'\|. \quad (\text{A.7})$$

**Proof**

Due to Proposition A.1  $\Phi^*$  is symmetric, and then continuous at the vertex. So that it is sufficient to prove (A.7). Let  $\Phi \in \mathcal{E}$  be positive and take  $\Phi_n \in \mathcal{PL}$  such that  $\Phi_n \rightarrow \Phi$  in  $H^1(\mathcal{G})$ . Take also a positive test function  $\chi \in C_0^\infty(\mathcal{G})$ . Moreover, let  $0 = a_0 < a_1 < a_2 < \dots$  be the values such that  $\{g = a_i\}$  has strictly positive measure. Notice that  $g$  restricted to  $\tilde{D}_i^* = \{a_{i-1} < g < a_i\}$  is monotone and invertible by Lemma A.1. Monotonicity of  $g$  also implies that its derivative exists almost everywhere and is in  $L_{loc}^1(\mathbb{R}^+)$ . Then the following inequalities hold:

$$|(\chi, \Phi^{*'})| = - \sum_{j=1}^N \int_0^\infty \chi_j(y) g'(y) dy \quad (\text{A.8})$$

$$= - \sum_{j=1}^N \sum_{i \geq 0} \int_{\tilde{D}_i} \chi_j(y) g'(y) dy \quad (\text{A.9})$$

$$= \sum_{j=1}^N \int_0^{g(0)} \chi_j(y^*(s)) ds \quad (\text{A.10})$$

$$\leq \liminf \sum_{j=1}^N \int_0^{g_n(0)} \chi_j(y_n^*(s)) ds \quad (\text{A.11})$$

$$= \liminf_n -(\chi, \Phi_n^{*'}) \quad (\text{A.12})$$

$$= \liminf_n |(\chi, \Phi_n^{*'})| \quad (\text{A.13})$$

$$\leq \liminf_n \|\chi\| \|\Phi_n^{*'}\| \quad (\text{A.14})$$

$$\leq \frac{N}{2} \liminf_n \|\chi\| \|\Phi_n'\| \quad (\text{A.15})$$

$$= \frac{N}{2} \|\chi\| \|\Phi'\|. \quad (\text{A.16})$$

The chain of inequalities stands for the following reasons. In (A.8) we explicitly wrote the r.h.s.. In (A.9) we have restricted the integral to the regions where  $g$  is not constant. In (A.10) we have changed variable of integration, the new one being  $y^*(s)$  defined as before. This is possible due to the restriction made in (A.9). In (A.11) we have used Fatou's Lemma and Lemma A.2. In (A.12) we have changed back the integration variable and in (A.13) we just changed a sign. In (A.14) we have used Cauchy-Schwarz inequality. In (A.15) we have used lemma A.2. In (A.16) we have used the convergence hypothesis. Estimate (A.7) for a positive function  $f$  follows from equation (A.16) by Riesz Theorem.

Now we extend the inequality to the general case. First notice that Proposition A.1 and Proposition A.2 both extend to the non-positive case and to the complex valued case. A careful inspection of the argument used above, shows that for positive  $\chi \in C_0^\infty(\mathcal{G})$  it is still valid until inequality (A.14). Then to conclude the proof we have to extend Proposition A.2 to complex valued functions.

For the real valued case, the extension is trivial. If  $\Phi$  is piecewise linear then  $|\Phi| \in \mathcal{PL}$  and has the same rearrangement of  $\Phi$ . Notice also that  $|\Phi'| = \|\Phi'\|$  almost everywhere. Therefore since (A.4) holds for  $|\Phi|$ , then it holds also for  $\Phi$ .



Now for the complex valued case, to define the class of approximating functions we set  $\Phi = e^{i\Theta}F$  with  $\Theta, F \in \mathcal{PL}$ , and where the product must be understood componentwise  $e^{i\Theta}F = (e^{i\theta_1}f_1, \dots, e^{i\theta_N}f_N)^T$ . This set is still dense in  $\mathcal{E}$ . Again notice that  $\Phi^* = F^*$ . We have also

$$|\Phi'|^2 = |\Theta'F|^2 + |F'|^2.$$

Therefore we can write

$$\|\Phi^{*'}\|^2 = \|F^{*'}\|^2 \leq \|F'\|^2 \leq \|\Theta'F\|^2 + \|F'\|^2 = \|\Phi'\|^2$$

which proves equation (A.7) in the general case. □

**Remark A.1.** *The same argument used to prove Theorem 6 can be used for the  $W^{1,p}$  norm since  $z \rightsquigarrow |z|^p$  is convex for  $p \geq 1$ .*

**Remark A.2.** *The constant  $N^2/4$  is optimal. For instance take  $\Phi$  such that*

$$\phi_1(y) = \begin{cases} x & 0 \leq x \leq 1 \\ 2 - x & 1 \leq x \leq 2 \\ 0 & x \geq 2 \end{cases} \quad \phi_i = 0 \text{ for } i \neq 1$$

*Then  $g$  can be easily computed by using (A.3). One has:*

$$g(x) = \begin{cases} 1 - \frac{N}{2}x & 0 \leq y \leq \frac{2}{N} \\ 0 & x \geq \frac{2}{N} \end{cases}$$

*From which*

$$\|\Phi'\|^2 = 2 \quad \|\Phi^{*'}\|^2 = \frac{N^2}{2}.$$

## APPENDIX B. USEFUL IDENTITIES

In this section we recall some useful identities that will be used several times in the paper. We label the soliton profile on the real line as

$$\phi_s(x) = [(\mu + 1)\omega]^{\frac{1}{2\mu}} \operatorname{sech}^{\frac{1}{\mu}}(\mu\sqrt{\omega}x). \quad (\text{B.1})$$

It satisfies the equation

$$-\phi_s'' - |\phi_s|^{2\mu}\phi_s = -\omega\phi_s. \quad (\text{B.2})$$

Moreover, multiplying by  $\bar{\phi}_s$  and integrating one checks that

$$\|\phi_s'\|_{L^2(\mathbb{R})}^2 - \|\phi_s\|_{L^{2\mu+2}(\mathbb{R})}^{2\mu+2} + \omega\|\phi_s\|_{L^2(\mathbb{R})}^2 = 0. \quad (\text{B.3})$$

Starting from definition (B.1) and changing variables in the integrals, one obtains the following formulas:

$$\int_0^\infty |\phi_s(x + \xi)|^2 dx = \frac{(\mu + 1)^\frac{1}{\mu}}{\mu} \omega^{\frac{1}{\mu} - \frac{1}{2}} \int_{\tanh(\xi\mu\sqrt{\omega})}^1 (1 - t^2)^{\frac{1}{\mu} - 1} dt \quad (\text{B.4})$$

$$\int_0^\infty |\phi_s(x + \xi)|^{2\mu+2} dx = \frac{(\mu + 1)^{1 + \frac{1}{\mu}}}{\mu} \omega^{\frac{1}{\mu} + \frac{1}{2}} \int_{\tanh(\xi\mu\sqrt{\omega})}^1 (1 - t^2)^{\frac{1}{\mu}} dt. \quad (\text{B.5})$$

## REFERENCES

- [1] R. Adami, C. Cacciapuoti, D. Finco, and D. Noja, *Constrained energy minimization for the focusing NLS on star graphs*, in preparation (2012).
- [2] R. Adami, C. Cacciapuoti, D. Finco, and D. Noja, *Fast solitons on star graphs*, Rev. Math. Phys. **23** (2011), no. 4, 409–451.
- [3] R. Adami, C. Cacciapuoti, D. Finco, and D. Noja, *On the structure of critical energy levels for the cubic focusing NLS on star graphs*, J. Phys. A: Math. Theor. **45** (2012), no. 49, 192001, 7pp.
- [4] R. Adami, C. Cacciapuoti, D. Finco, and D. Noja, *Stationary states of NLS on star graphs*, arXiv:1104.3839 (2011), 4pp.
- [5] R. Adami and D. Noja, *Existence of dynamics for a 1D NLS equation perturbed with a generalized point defect*, J. Phys. A: Math. Theor. **42** (2009), no. 49, 495302, 19pp.
- [6] R. Adami and D. Noja, *Stability and symmetry breaking bifurcation for the ground states of a NLS equation with a  $\delta'$  interaction*, arXiv:1112.1318, to appear on Commun. Math. Phys.
- [7] R. Adami, D. Noja, and A. Sacchetti, *Bose-Einstein condensates: Theory, characteristics, and current research*, Nova Publishing, New York, 2010.
- [8] R. Adami, D. Noja, and N. Visciglia, *Constrained energy minimization and ground states for NLS with point defects*, arXiv:1204.6344, 2012, submitted.
- [9] S. Albeverio, F. Gesztesy, R. Høgh-Krohn, and H. Holden, *Solvable models in quantum mechanics: Second edition*, AMS Chelsea Publ., 2005, with an Appendix by P. Exner.
- [10] G. Berkolaiko, R. Carlson, S. Fulling, and P. Kuchment, *Quantum graphs and their applications*, Contemporary Math., vol. 415, American Math. Society, Providence, R.I., 2006.
- [11] J. Blank, P. Exner, and M. Havlicek, *Hilbert spaces operators in quantum physics*, Springer, New York, 2008.
- [12] H. Brezis and E. H. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983), 486–490.
- [13] D. Cao Xiang and A. B. Malomed, *Soliton defect collisions in the nonlinear Schrödinger equation*, Phys. Lett. A **206** (1995), 177–182.
- [14] T. Cazenave, *Semilinear Schrödinger Equations*, Courant Lecture Notes in Mathematics, AMS, vol. **10**, Providence, 2003.
- [15] T. Cazenave and P.-L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Commun. Math. Phys. **85** (1982), 549–561.
- [16] J. Fröhlich, S. Gustafson, B. L. G. Jonsson, and I. Sigal, *Solitary wave dynamics in an external potential*, Comm. Math. Phys. **250** (2004), 613–642.
- [17] R. Fukuizumi, M. Ohta, and T. Ozawa, *Nonlinear Schrödinger equation with a point defect*, Ann. Inst. H. Poincaré Anal. Non Linéaire **25** (2008), no. 5, 837–845.
- [18] S. Gnutzman, U. Smilansky, and S. Derevyanko, *Stationary scattering from a nonlinear network*, Phys. Rev. A **83** (2011), 033831.
- [19] M. Grillakis, J. Shatah, and W. Strauss, *Stability theory of solitary waves in the presence of symmetry I*, J. Funct. Anal. **74** (1987), 160–197.
- [20] M. Grillakis, J. Shatah, and W. Strauss, *Stability theory of solitary waves in the presence of symmetry II*, J. Funct. Anal. **94** (1990), 308–348.
- [21] K. Hildén, *Symmetrization of functions in Sobolev spaces and the isoperimetric inequality*, Manuscripta Math. **18** (1976), no. 3, 215–235.
- [22] B. Kawohl, *Rearrangements and convexity of level sets in PDE*, Lecture Notes in Mathematics, vol. 1150, Springer-Verlag, Berlin, 1985.
- [23] V. Kostrykin and R. Schrader, *Kirchhoff's rule for quantum wires*, J. Phys. A: Math. Gen. **32** (1999), no. 4, 595–630.
- [24] P. Kuchment, *Quantum graphs. I. Some basic structures*, Waves Random Media **14** (2004), no. 1, S107–S128.
- [25] P. Kuchment, *Quantum graphs. II. Some spectral properties of quantum and combinatorial graphs*, J. Phys. A: Math. Gen. **38** (2005), no. 22, 4887–4900.
- [26] S. Le Coz, R. Fukuizumi, G. Fibich, B. Ksherim, and Y. Sivan, *Instability of bound states of a nonlinear Schrödinger equation with a Dirac potential*, Phys. D **237** (2008), no. 8, 1103–1128.

- [27] E. H. Lieb and M. Loss, *Analysis*, second ed., Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 2001.
- [28] Y. Linzon, R. Morandotti, M. Volatier, V. Aimez, R. Ares, and S. Bar-Ad, *Nonlinear scattering and trapping by local photonic potentials*, Phys. Rev. Lett. **99** (2007), 133901.
- [29] A. E. Miroshnichenko, M. I. Molina, and Y. S. Kivshar, *Localized modes and bistable scattering in nonlinear network junctions*, Phys. Rev. E **75** (2007), 046602.
- [30] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Inequalities involving functions and their integrals and derivatives*, Mathematics and Its Applications, vol. 53, Kluwer Academic Publishers, Dordrecht/Boston/London, 1991.
- [31] M. Peccianti, A. Dyadyusha, M. Kaczmarek, and G. Assanto, *Escaping solitons from a trapping potential*, Phys. Rev. Lett. **101** (2008), 153902.
- [32] Z. Sobirov, D. Matrasulov, K. Sabirov, S. Sawada and K. Nakamura *Integrable nonlinear Schrödinger equation on simple networks: connection formula at vertices*, Phys. Rev. E **81** (2010), 066602.
- [33] A. Tokuno, M. Oshikawa, and E. Demler, *Dynamics of the one dimensional Bose liquids: Andreev-like reflection at Y-junctions and the absence of Aharonov-Bohm effect*, Phys. Rev. Lett. **100** (2008), 140402.
- [34] W. Wan, S. Muenzel, and J. W. Fleischer, *Wave tunneling and hysteresis in nonlinear junctions*, Phys. Rev. Lett. **104** (2010), 073903.
- [35] M. Weinstein, *Modulational stability of ground states of nonlinear Schrödinger equations*, SIAM J. Math. Anal. **16** (1985), 472–491.
- [36] M. Weinstein, *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure Appl. Math **39** (1986), 51–68.

DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, UNIVERSITÀ DI MILANO BICOCCA, VIA R. COZZI, 53, 20125 MILANO, ITALY

*E-mail address:* `riccardo.adami@unimib.it`

HAUSDORFF CENTER FOR MATHEMATICS, INSTITUT FÜR ANGEWANDTE MATHEMATIK, ENDENICHER ALLEE, 60, 53115 BONN, GERMANY

*E-mail address:* `cacciapuoti@hcm.uni-bonn.de`

FACOLTÀ DI INGEGNERIA, UNIVERSITÀ TELEMATICA INTERNAZIONALE UNINETTUNO, CORSO VITTORIO EMANUELE II, 39, 00186 ROMA, ITALY

*E-mail address:* `d.finco@uninettunouniversity.net`

DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, UNIVERSITÀ DI MILANO BICOCCA, VIA R. COZZI, 53, 20125 MILANO, ITALY

*E-mail address:* `diego.noja@unimib.it`